

G6004: Problem Set #5

Due: Tuesday, November 28

Problem 1: Thermal pulses and the thin-shell instability

Evolved stars with nuclear burning in thin shells often exhibit a long-term (thousands of years) pulsation in the nuclear burning rate. This problem explores the origin of this instability.

A. First, let's demonstrate that core burning is stable. In hydrostatic equilibrium, show that a small perturbation in the central pressure (dP_c) gives rise to a (small) change in the central density ($d\rho_c$) as given by

$$\frac{dP_c}{P_c} = \frac{4}{3} \frac{d\rho_c}{\rho_c}$$

(hint: review the section on polytropes, and argue that the core can reasonably be described by a polytrope).

Next, using the ideal gas law, re-write this as a relation between the central temperature perturbation (dT_c) and the resulting central density perturbation ($d\rho_c$). Show that these have the same sign (i.e. compression leads to heating while expansion leads to cooling) and argue that this is a necessary (but not necessarily sufficient) condition for stability in a star's nuclear burning region.

B. Now we will repeat this exercise for a thin nuclear-burning shell of thickness $l = r - r_0$, where l is much smaller than either r , the outer boundary, or r_0 , the fixed inner boundary. Imagine for a moment that the rate of nuclear burning exceeds the net heat flow out of the shell. The shell will expand, increasing r and lifting the layers above it. Show that, in hydrostatic equilibrium, the resulting pressure perturbation (dP) within the shell (which contains a fixed amount of mass) is given by

$$\frac{dP}{P} = -4 \frac{dr}{r}.$$

C. The shell's (fixed) mass is given by $\Delta m = 4\pi r_0^2 l \rho$. Using this and the previous result, show that

$$\frac{d\rho}{\rho} = -\frac{dr}{r} \frac{r}{l}$$

D. Finally, use the ideal gas law and the same reasoning above for stability (i.e. the temperature perturbation must have the same sign as the density perturbation) to find a stability condition:

$$4 \frac{l}{r} < 1$$

This shows that if the shell is thin enough (i.e. l/r is small enough), the shell may be unstable.

Problem 2: Linear, adiabatic pulsations of a homogeneous sphere

The spherically-symmetric, time-dependent equations of hydrodynamics can be used to derive an equation for the evolution of a small perturbation of the form:

$$r(M_r, t) = r_0(M_r) \left[1 + x(M_r) e^{i\omega t} \right]$$

where the unperturbed star has pressure $P_0(M_r)$, density $\rho_0(M_r)$ and position $r_0(M_r)$. Here M_r is the usual radial mass shell coordinate and we assume that $x \ll 1$. In class we sketched out how to derive an eigenvalue equation:

$$\frac{d^2 x}{dr_0^2} + \left(\frac{4}{r_0} - \frac{\rho_0 g_0}{P_0} \right) \frac{dx}{dr_0} + \frac{\rho_0}{\gamma_{\text{ad}} P_0} \left[\omega^2 + (4 - 3\gamma_{\text{ad}}) \frac{g_0}{r_0} \right] x = 0$$

where $g_0 = GM_r/r_0^2$ is the gravitational acceleration of the unperturbed star. We can take this as our starting place (although I encourage you to work through the steps in deriving this equation).

A. For a unperturbed star with constant density ρ , mass M and radius R , we can easily solve the (time-independent) equation of hydrostatic equilibrium to find $P_0(r_0)$ and $g_0(r_0)$. Do this, and show that the above equation can be simplified to

$$\frac{d^2 x}{d\xi^2} + \left(\frac{4}{\xi} - \frac{2\xi}{1 - \xi^2} \right) \frac{dx}{d\xi} + \frac{\tilde{A}}{1 - \xi^2} x = 0$$

where we have defined two new dimensionless variables:

$$\tilde{A} = \frac{3\omega^2}{2\pi G \rho_0 \gamma_{\text{ad}}} + \frac{2(4 - 3\gamma_{\text{ad}})}{\gamma_{\text{ad}}}$$

and $\xi = r_0/R_0$.

B. Demonstrate that the fundamental mode solution is $x = \text{constant}$ and derive the resulting eigenfrequency ω_0 , showing that it implies ω_0^2/ρ_0 is a constant. This is the same period-density relation shown by oscillating polytropes and is at the root of the period-luminosity relation of cepheids.

C. Attempt to find the first overtone, which is a second-order polynomial in ξ (i.e. $x = a + b\xi + c\xi^2$), and show that its eigenfrequency is given by:

$$\omega_1^2 = \frac{4\pi}{3} G \rho_0 (3\gamma_{\text{ad}} - 4) \left(1 + \frac{7\gamma_{\text{ad}}}{3\gamma_{\text{ad}} - 4} \right)$$