Physics 8012
Problem set 1, due on 9/10/08
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In this problem, you are to show that the deflection angle of a photon by a point mass M is $4GM/b$, where b is the impact parameter (setting $c = 1$ as usual). This is such a well known result that you can find it in any GR textbook. However, don't go look! This is something you should be able to figure out by following the steps below.

First, let's start with a metric of the following form

$$ds^2 = -(1+2\phi)d\eta^2 + (1-2\phi)\delta_{ij}dx^i dx^j \tag{1}$$

Let's assume ϕ is small and so are all its derivatives, and we will be essentially doing first order perturbation theory i.e only keep terms linear in ϕ or its derivatives. Let us further assume that ϕ is a function of position but not of time (anticipating that eventually we will set ϕ to be equal to the gravitational potential of a point mass).

a. Within the approximation of first order perturbation theory, show that the non-zero components of the affine connection are:

$$\begin{aligned} \Gamma^0{}_{0i} = \Gamma^0{}_{i0} = \Gamma^i{}_{00} &= \phi_{,i} \\ \Gamma^i{}_{jk} &= -\delta_{ij}\phi_{,k} - \delta_{ik}\phi_{,j} + \delta_{jk}\phi_{,i} \end{aligned} \tag{2}$$

where $\phi_{,i}$ means $\partial_i \phi$. Here, 0 is the time component and i denotes one of the spatial components. The above expressions are correct to first order in perturbation (i.e. treating ϕ and its derivatives as a small quantity). Note that I have not been very careful about the upstairs/downstairs placement of the spatial indices (Latin indices like i, j and k). To the lowest order, there is no difference between upstairs and downstairs spatial indices.

b. Write down the geodesic equations for both the time and the spatial components. Use λ as the affine parameter (i.e. that which parametrizes the photon trajectory). These equations should have both zero order and first order terms (zero order means independent of ϕ and its derivatives).

c. Let us write the solution to the above equations as $\eta(\lambda) = \bar{\eta}(\lambda) + \delta\eta(\lambda)$ and $x^i(\lambda) = \bar{x}^i(\lambda) + \delta x^i(\lambda)$, where $\bar{\eta}$ and $\bar{x}^i$ are the zero order pieces and the rest are the first order pieces. Show that $\bar{\eta}(\lambda) = \lambda$ and $\bar{x}^i(\lambda) = x_0^i + \lambda \hat{n}^i$,where x_0^i is some constant vector and $\hat{n}^i$ is some unit vector, satisfies the zero order geodesic equations. Check that this solution is null. You can think of $\bar{\eta}(\lambda)$ and $\bar{x}^i(\lambda)$ as describing the unperturbed path i.e. this is the path of the photon if ϕ were zero.

d. Next, solve the *spatial* geodesic equation for $d\delta x^i/d\lambda$. You will find that the solution involves an integral of derivatives of ϕ over λ. Note that $\phi = \phi(\mathbf{x}(\lambda))$ is a function of the photon trajectory. A key approximation we will make for the next part of this problem is that one can approximate $\phi(\mathbf{x}(\lambda)) \sim \phi(\bar{\mathbf{x}}(\lambda))$, because any difference is second order (remember that ϕ itself is already first order).

e. Finally, let's assume that ϕ is due to a point mass M sitting at the origin i.e. $\phi = -GM/r$ where r is the distance from point mass to photon. Let's also assume that the unit vector $\hat{\mathbf{n}}$ points in the x direction i.e. $\hat{n}^1 = 1, \hat{n}^2 = 0, \hat{n}^3 = 0$, and let's assume $x_0^1 = 0$, $x_0^2 = b$, $x_0^3 = 0$ (i.e. b is the impact parameter). Use your result from **d.** to complete the derivation for the deflection angle.

f. Estimate the angle of deflection by a galaxy of mass $10^{12} M_\odot$ (treating it as a point mass) and an impact parameter of 10 kpc. These numbers are fairly typical of the gravitational lenses we observe in nature. Note that $M_\odot$ means a solar mass. Express the angle in arc seconds.