Physics 8012
Problem set 6, due on 10/29/08
Lam Hui

In this problem set, I ask you to check and extend some of the calculations done in the class on gravitational wave production and emission.

Let's start with Carroll's equation 7.153:

$$R^{(2)}_{\mu\nu} = \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} + \frac{1}{4}(\partial_\mu h_{\rho\sigma})\partial_\nu h^{\rho\sigma} + (\partial^\sigma h^\rho{}_\nu)\partial_{[\sigma}h_{\rho]\mu} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} \tag{1}$$
$$+\frac{1}{2}\partial_\sigma(h^{\rho\sigma}\partial_\rho h_{\mu\nu}) - \frac{1}{4}(\partial_\rho h_{\mu\nu})\partial^\rho h - (\partial_\sigma h^{\rho\sigma} - \frac{1}{2}\partial^\rho h)\partial_{(\mu}h_{\nu)\rho}$$

I have checked this and it seems to be correct. You are welcome to check it yourself, though I won't ask you to do it here. Using this as a given, and using the definition for the gravitational wave energy-momentum tensor:

$$t_{\mu\nu} \equiv -\frac{1}{8\pi G}\left[R^{(2)}_{\mu\nu} - \frac{1}{2}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}\eta_{\mu\nu}\right] \tag{2}$$

verify that in harmonic gauge, you do obtain the result we got in class:

$$\langle t_{\mu\nu}\rangle = \frac{1}{32\pi G}\langle\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma} - \frac{1}{2}\partial_\mu h\partial_\nu h\rangle \tag{3}$$

Show that this is *inconsistent* with Carroll's equation 7.165 which is supposedly valid in any gauge.

In class, we then went on to obtain, again in harmonic gauge, the energy flux

$$t_{0k}n^k = \frac{-G}{8\pi r^2}\langle\frac{d^3I_{ij}}{dt^3}\frac{d^3I_{pq}}{dt^3}\rangle\left[\delta^{ip}\delta^{jq} - 2n^in^p\delta^{qj} + \frac{1}{2}n^in^jn^pn^q - \frac{1}{2}\delta^{ij}\delta^{pq} + \frac{1}{2}n^in^j\delta^{pq} + \frac{1}{2}\delta^{ij}n^pn^q\right] \tag{4}$$

where r is the distance to source, and n^k is the unit radial vector. The power emitted per solid angle is related to the energy flux by

$$\frac{dP}{d\Omega} = t_{0k}n^kr^2 \tag{5}$$

Using the definition

$$J_{ij} \equiv I_{ij} - \frac{1}{3}\delta_{ij}I^k{}_k \tag{6}$$

show that

$$\frac{dP}{d\Omega} = -\frac{G}{8\pi}\langle\frac{d^3J_{ij}}{dt^3}\frac{d^3J_{pq}}{dt^3}\rangle\left[\delta^{ip}\delta^{jq} - 2\delta^{ip}n^jn^q + \frac{1}{2}n^in^jn^pn^q\right] \tag{7}$$

This agrees with Carroll's equation 7.187. Notice that we have obtained this without going through transverse traceless gauge at all.

Verify that once you integrate over solid angle, you obtain also Carroll's result:

$$P = -\frac{G}{5}\langle\frac{d^3J_{ij}}{dt^3}\frac{d^3J^{ij}}{dt^3}\rangle \tag{8}$$

Let's apply equation (7) to our favorite binary system and work out exactly the angular dependence of the radiation power.

Recall that for an equal mass binary system, the quadrupole moment components are:

$$\begin{aligned} I_{11} = MR^2(1 + \cos 2\Omega t) \quad , \quad I_{22} = MR^2(1 - \cos 2\Omega t) \\ I_{12} = I_{21} = MR^2 \sin 2\Omega t \quad , \quad I_{13} = I_{31} = I_{23} = I_{32} = I_{33} = 0 \end{aligned} \tag{9}$$

where M is the mass of each star, R is the distance from center of mass, $\Omega = (GM/4R^3)^{1/2}$, and the two stars lie in the x-y plane.

Work out J_{ij}, and then using $(n^1, n^2, n^3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, show that

$$\frac{dP}{d\Omega} = -\frac{16}{\pi} G M^2 R^4 \Omega^6 \left[1 - \sin^2\theta + \frac{1}{8}\sin^4\theta\right] \tag{10}$$

Interestingly, if this is correct (which you should check!), the power does not vanish in the z direction, nor in the x-y plane, but somewhere in between. Compare it with the angular distribution of the dipole radiation you are familiar with in E & M.

Lastly, integrate over angle and see if you can reproduce Carroll's equation 7.192.