Physics 8012
Problem set 7, due on 11/17/08
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In this problem set, you are asked to verify that the appropriate gauge transformation from a general harmonic gauge to the transverse traceless (TT) gauge is accomplished by the projecting operation defined by Carroll.

We are interested in a plane wave traveling in vacuum. In harmonic gauge, the solution to $\Box\bar{h}_{ij} = 0$ is

$$\bar{h}_{ij} = C_{ij} e^{ik\cdot x} \tag{1}$$

where C_{ij} is a constant matrix, $k \cdot x \equiv k_\mu x^\mu$, $k^2 \equiv k_\mu k^\mu = 0$, and $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$. First, show that by the harmonic gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$, we have

$$\bar{h}_{0j} = -n_i \bar{h}^i{}_j \quad , \quad \bar{h}_{00} = n_i n_j \bar{h}^{ij} \tag{2}$$

where $n_i \equiv k_i/\omega$, with ω being the energy of the graviton (i.e. $k \cdot x = -\omega t + k_i x^i$).

From this, show that

$$\begin{aligned} h_{00} &= \frac{1}{2}(n_i n_j + \delta_{ij})\bar{h}^{ij} \\ h_{0j} &= -n_i \bar{h}^i{}_j \\ h_{ij} &= (\delta_{ip}\delta_{jq} + \frac{1}{2}\delta_{ij} n_p n_q - \frac{1}{2}\delta_{ij}\delta_{pq})\bar{h}^{pq} \end{aligned} \tag{3}$$

Note that even if $\bar{h}^{pq}$ were independent of direction (i.e. n^i), as in the case of the quadrupole formula, h_{ij} is direction dependent.

To transform to TT gauge, we need to find ξ_μ such that

$$h^{TT}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{4}$$

where $h_{\mu\nu}$ on the right is in general harmonic gauge, and $h^{TT}_{\mu\nu}$ on the left is in TT gauge i.e. $h^{TT}_{00} = h^{TT}_{0i} = h^{TT}_{00} = k^i h^{TT}_{ij} = 0$.

Show that

$$\begin{aligned} \xi_0 &= \frac{1}{4i\omega}(\delta_{ij} + n_i n_j)\bar{h}^{ij} \\ \xi_i &= \frac{1}{i\omega}\left(-n_p\delta_{qi} + \frac{1}{4}n_i\delta_{pq} + \frac{1}{4}n_p n_q n_i\right)\bar{h}^{pq} \end{aligned} \tag{5}$$

would do the job.

Finally, show that

$$h^{TT}_{ij} = \left(P_{ip}P_{jq} - \frac{1}{2}P_{ij}P_{pq}\right)\bar{h}^{pq} \tag{6}$$

where P is exactly the projection operator used by Carroll:

$$P_{ij} \equiv \delta_{ij} - n_i n_j \tag{7}$$