

Physics 8012
Problem set 8, due on 11/17/08
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In this problem set, you are asked to work out the Gibbons-Hawking boundary term as part of the path integral derivation of Hawking radiation. You are also asked to carry out a third derivation of Hawking radiation, following a strategy developed by Parikh and Wilczek.

1. Recall that we can write the partition function Z as

$$Z \sim e^{-S_E(\text{classical})} \quad (1)$$

where $S_E(\text{classical})$ is the Euclidean action evaluated at the classical path, or more accurately, the classical field configuration (i.e. the Schwarzschild metric in our case). Note that the above ignores higher order quantum corrections i.e. if we had kept track of \hbar , $Z = \exp[-S_E(\text{classical})/\hbar](1 + O(\hbar))$. S_E has two contributions, one is the usual Einstein-Hilbert term, the other is the Gibbons-Hawking boundary term:

$$-S_E = i \int d^4x \sqrt{-g} R + 2i \int d\Sigma[\kappa] \quad (2)$$

For the Schwarzschild metric, $R = 0$. We need only worry about the surface term, and we will evaluate it at a surface of constant r , taking the limit $r \rightarrow \infty$. Therefore, $d\Sigma = dt d\theta d\phi \sqrt{(1 - 2GM/r)r^4 \sin^2\theta}$, where the square root comes from $\sqrt{-h}$ where h is the determinant of the (t, θ, ϕ) part of the metric:

$$h_{tt} = -(1 - 2GM/r) \quad , \quad h_{\theta\theta} = r^2 \quad , \quad h_{\phi\phi} = r^2 \sin^2\theta \quad (3)$$

Using $t_E = it$, we can rewrite $i d\Sigma = dt_E d\theta d\phi \sqrt{(1 - 2GM/r)r^4 \sin^2\theta}$. The symbol $[\kappa]$ denotes the extrinsic curvature for a Schwarzschild metric minus the extrinsic curvature for a Minkowski metric (think of this as the $M = 0$ limit of the Schwarzschild metric). The extrinsic curvature for our metric is given by

$$\kappa = h^{ij} \kappa_{ij} \quad , \quad \kappa_{ij} = \frac{1}{2N} \frac{\partial h_{ij}}{\partial r} \quad (4)$$

where $N = 1/\sqrt{g^{rr}} = (1 - 2GM/r)^{-1/2}$. Note that our definition for κ_{ij} is not general, but works for a diagonal metric such as ours (with no mixed metric terms between r and the other coordinates). Recalling that t_E is periodic with a period of $8\pi GM$, show that

$$Z \sim e^{-32\pi^2 G^2 M^2} \quad (5)$$

Equating this with $e^{-F/T}$ where F is the free energy and $T = 1/(8\pi GM)$ is what gives us the entropy law of $S = \text{Area}/(4G)$, where $\text{Area} = 4\pi(2GM)^2$.

2. Here, you are asked to work out the derivation due to Parikh and Wilczek (hep-th/9907001). Let's start with the Schwarzschild metric:

$$ds^2 = -(1 - 2GM/r) dt_S^2 + (1 - 2GM/r)^{-1} dr^2 + r^2 d\Omega^2 \quad (6)$$

We call the Schwarzschild time here t_S instead of the usual t in anticipation of a coordinate transformation we are going to do:

$$t = t_S + 2\sqrt{2GM r} + 2GM \ln \left[\frac{\sqrt{r} - \sqrt{2GM}}{\sqrt{r} + \sqrt{2GM}} \right] \quad (7)$$

Show that the metric becomes

$$ds^2 = -dt^2 + \left(\sqrt{\frac{2GM}{r}} dt + dr \right)^2 + r^2 d\Omega^2 \quad (8)$$

This is called the Painlevé metric.

Show that the radial null geodesic is given by

$$\dot{r} = \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2GM}{r}} \quad (9)$$

We will be interested in Hawking radiation that eventually escapes to infinity, hence we will take the positive sign.

More precisely, we are interested in a black hole of mass M emitting a (spherically symmetric) shell of mass ω . The correct motion of such a shell is described by a geodesic in the modified metric:

$$ds^2 = -dt^2 + \left(\sqrt{\frac{2G(M-\omega)}{r}} dt + dr \right)^2 + r^2 d\Omega^2 \quad (10)$$

such that

$$\dot{r} = \frac{dr}{dt} = 1 - \sqrt{\frac{2G(M-\omega)}{r}} \quad (11)$$

That such a description correctly describes the backreaction of ω on the background geometry is supposedly proven in Kraus & Wilczek (gr-qc/9408003). Though it seems reasonable, it is perhaps a little surprising that this description is supposed to work even when ω is comparable to M .

The next step in the derivation is to recall something you have learned in your quantum mechanics class: the WKB description of a quantum tunneling process tells you that the tunneling probability is given by

$$\Gamma \sim e^{-2 \int dx \sqrt{2(V-E)}} \quad (12)$$

where V is the potential, E is the energy, and x is integrated over the classically forbidden region. Recalling that that $E = \dot{x}^2/2 + V$, one can see that

$$\Gamma \sim e^{-2 \text{Im} \int dx \dot{x}} \quad (13)$$

Applying this prescription to our case (tunneling of a spherical shell out of a black hole), we have

$$\Gamma \sim e^{-2 \text{Im} \int dr p_r} \quad (14)$$

where p_r is the radial momentum of our shell of mass ω i.e. we are interested in

$$I \equiv \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} p_r dr = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^{p_r} dp'_r dr \quad (15)$$

The Hamiltonian equation of motion $\dot{r} = dH/dp_r$ tells us that

$$I = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_M^{M-\omega} \frac{dH}{\dot{r}} dr \quad (16)$$

where we have assumed that $p_r = 0$ at $H = M$.

Let's do this integral two different ways. First, exchanging the order of integration, and setting $H = M - \omega'$, we have

$$I = -\text{Im} \int_0^\omega \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{dr}{1 - \sqrt{\frac{2G(M-\omega')}{r}}} d\omega' \quad (17)$$

We have in mind r_{in} is just inside the horizon of the original black hole of mass M , and r_{out} is just outside the horizon of the diminished black hole of mass $M - \omega$. Therefore, $r_{\text{in}} < r_{\text{out}}$! The integral over r therefore passes through a pole at $r = 2G(M - \omega')$ (recall that ω' is between 0 and ω).

Using standard procedures in complex analysis, show that

$$\text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{dr}{1 - \sqrt{\frac{2G(M-\omega')}{r}}} = \pm 4\pi G(M - \omega) \quad (18)$$

where the sign depends on whether you go over or under the pole in the complex r plane. We will choose the $-$ sign (corresponding to taking the contour into the lower r plane, keeping in mind that $r_{\text{in}} > r_{\text{out}}$). This choice is dictated by the desire to have a negative I - tunneling probability should always be suppressed.

Hint: you should be able to relate your problem to doing an integral like $\int dz/(z - A)$ which can be decomposed into 3 parts: one involves an integration of z up to $A - \epsilon$, the other involves an integration of z from $A - \epsilon$ to $A + \epsilon$, and the third piece involves an integration from $A + \epsilon$ onward. The first and third pieces are real. The second piece can be done along a semicircle in the upper z or lower z plane, yielding $\pm i\pi$.

Therefore, show that

$$I = 4\pi G\omega(M - \omega/2) \quad (19)$$

We can also do the integral in a different way by integrating over $M' = M - \omega'$ first:

$$I = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_M^{M-\omega} \frac{dM'}{1 - \sqrt{\frac{2GM'}{r}}} dr \quad (20)$$

Using similar arguments as before, show that

$$\text{Im} \int_M^{M-\omega} \frac{dM'}{1 - \sqrt{\frac{2GM'}{r}}} = \pm \pi r/G \quad (21)$$

We will choose the $-$ sign which corresponds to taking M' into the lower plane. This choice gives the right positive I once we integrate r from $r_{\text{in}} = 2GM$ to $r_{\text{out}} = 2G(M - \omega)$. Verify that we obtain once again $I = 4\pi G\omega(M - \omega/2)$.

To conclude, we find that the tunneling probability is

$$\Gamma \sim e^{-2I} = e^{-8\pi G\omega(M-\omega/2)} = e^{\delta \text{entropy}} \quad (22)$$

where $\delta \text{entropy} = 4\pi G(M-\omega)^2 - 4\pi GM^2$. Therefore, this tunneling calculation is consistent with the entropy law we derived earlier using other methods. Interestingly, this tunneling calculation also shows that Hawking radiation is not entirely thermal. See Parikh and Wilczek for further discussion. You might also wonder why we have no discussions of the vacuum state in this derivation. Indeed, a particular vacuum state is secretly assumed here. See discussion in Kraus and Wilczek (gr-qc/9406042).