Columbia University Physics G8050, Spring 2005
Advanced Mathematical Methods –
Introduction to String Theory

A Short Note on the Non-relativistic Limit of the Nambu-Goto Action by
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In the lecture on Jan. 20, we rewrote the Nambu-Goto action in terms of the string transverse velocity, and then took its non-relativistic limit to check that it makes sense. The explanations given were confusing, and since this material is not included in Polchinski, let me write here a short note to give you the proper explanations (following those in Zwiebach). The points developed here are actually not central to the development of the main subject of the class, but they might be useful for getting some intuitive feel for the nature of the action.

The starting point is the Nambu-Goto action:

$$S_{\rm NG} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-\det[\partial_a X^\mu \partial_b X_\mu]} \tag{1}$$

This can be simply understood as the proper area (in the 'time' sense) swept out by the string in space-time, multiplied by the string tension with a negative sign.

Recall that we have freedom in the choice of τ and σ. Let us choose $\tau = X^0$ i.e. τ is just the time of the space-time. In that case, note that

$$\begin{aligned} \partial_\tau X^\mu &= (1, \partial_\tau \mathbf{X}) \\ \partial_\sigma X^\mu &= (0, \partial_\sigma \mathbf{X}) \end{aligned} \tag{2}$$

where we have used the bold-faced $\mathbf{X}$ to denote the *spatial* components of X^μ. Note that we have used $\partial_\sigma X^0 = 0$ which holds because $X^0 = \tau$ and by definition, ∂_σ is holding τ fixed.

Next, let us define a *spatial* length along the string at any particular fixed time τ:

$$ds = |d\mathbf{X}| = |\partial_\sigma \mathbf{X}| d\sigma \tag{3}$$

Now, let us define a transverse velocity in the *spatial* sense:

$$\mathbf{v} \equiv \partial_\tau \mathbf{X} - (\partial_\tau \mathbf{X} \cdot \partial_s \mathbf{X})\partial_s \mathbf{X} \tag{4}$$

Such a velocity makes sense: $\partial_\tau \mathbf{X}$ itself can be thought of as the velocity of a particular point on the string, but we are interested in the part of the velocity that is orthogonal to the string, and since $\partial_s \mathbf{X}$ is a unit vector tangent to the string (at any particular time τ), the above is indeed a transverse velocity. The fact that $\partial_s \mathbf{X}$ is tangent to the string follows from the fact that ∂_s is proportional to ∂_σ. The fact that $\partial_s \mathbf{X}$ is a unit vector follows from eq. (3).

With the above groundwork laid, we can proceed to rewrite $S_{\rm NG}$. Written out explicitly, it is

$$S_{\rm NG} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{-[(\partial_\tau X^\mu\partial_\tau X_\mu)(\partial_\sigma X^\mu\partial_\sigma X_\mu) - (\partial_\tau X^\mu\partial_\sigma X_\mu)^2]} \tag{5}$$
$$= -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{-[(-1+|\partial_\tau\mathbf{X}|^2)(|\partial_\sigma\mathbf{X}|^2) - (\partial_\tau\mathbf{X}\cdot\partial_\sigma\mathbf{X})^2]}$$

where we have made use of eq. (2).

Let us replace $\partial_\sigma\mathbf{X}$ by $(ds/d\sigma)\partial_s\mathbf{X}$, and use eq. (3) to set $|\partial_\sigma\mathbf{X}| = ds/d\sigma$. Therefore, the action is

$$S_{\rm NG} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma(ds/d\sigma)\sqrt{-[(-1+|\partial_\tau\mathbf{X}|^2) - (\partial_\tau\mathbf{X}\cdot\partial_s\mathbf{X})^2]} \tag{6}$$
$$= -\frac{1}{2\pi\alpha'}\int d\tau d\sigma(ds/d\sigma)\sqrt{1 - [|\partial_\tau\mathbf{X}|^2 - (\partial_\tau\mathbf{X}\cdot\partial_s\mathbf{X})^2]}$$
$$= -\frac{1}{2\pi\alpha'}\int d\tau ds\sqrt{1-|\mathbf{v}|^2}$$

Taking the non-relativistic limit is then simple: $|\mathbf{v}| \ll 1$ gives us

$$S_{\rm NG} \sim \int d\tau\left(-[\int ds/(2\pi\alpha')] + \frac{1}{2}[\int ds/(2\pi\alpha')|\mathbf{v}|^2]\right) \tag{7}$$

Such an action makes sense. $[\int ds/(2\pi\alpha')]$ gives the potential energy of the string at time τ, which is simply the length times tension of the string. The second term is obviously the kinetic energy, with each segment of the string carrying a kinetic energy of $ds/(2\pi\alpha')|\mathbf{v}|^2$. In other words, the segment can be thought of as having a mass of $ds/(2\pi\alpha')$. Or $1/(2\pi\alpha')$ is the mass per unit length of the string. Note the interesting feature that for the string we are interested in, the tension and the mass per unit length are one and the same. That our action respects Lorentz symmetry, when not taking the non-relativistic limit, demands this is so.