Physics G8050: Introduction to String Theory
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Problem Set 1, due Feb. 15, 2005

You might be worried that this problem set looks long. Well, don't you worry. Most of the the problem set consists of background explanations to help you better understand the materials i.e. it might take you longer to read the problem set than to do it! This is especially true in problem 2. To make it clear what it is that I would like you to work out, I use bold-faced words such as **show**, **check**, *etc to preface calculations that I would like you to do.*

1. Show that open string end-points move (classically) at the speed of light. One way to proceed is as follows: As we will show in the class later, it is always possible to choose coordinates such that the world-sheet metric is flat i.e. $\gamma_{\tau\tau} = -1, \gamma_{\sigma\sigma} = 1, \gamma_{\tau\sigma} = 0$. With such a gauge choice,

a. show that the open string boundary condition of $\partial^\sigma X^\mu = 0$ at $\sigma = 0, \ell$ is equivalent to $\partial_\sigma X^\mu = 0$ at $\sigma = 0, \ell$.

b. Making use of the above, **show** further that the vanishing of $T_{\sigma\sigma}$, the $\sigma\sigma$ component of the world-sheet energy momentum tensor (which follows from local world-sheet diffeomorphism invariance), implies that $\partial_\tau X^\mu \partial_\tau X_\mu = 0$ at the open string end-points i.e. $\partial_\tau X^\mu$ is null.

c. Show that the above statement, that the open string end-point moves at the speed of light, is in fact independent of coordinate choice, by arguing that $\partial_\tau X^\mu \propto t^a \partial_a X^\mu$, where t^a is tangent to the world-path of the open string end-point.

2. In this problem, you are to demonstrate an important result: that Lorentz symmetry is respected by the quantum open (bosonic) string theory only if $D = 26$. The full calculation is long and tedious. Don't worry, I won't make you do all of them. To make this problem set more manageable, I will supply most of the intermediate results without requiring proof on your part. You are of course welcome to check them. I will attempt here to sketch the path from results established in class to the all important final result of $D = 26$, giving you sufficient details so that, if you want to, you can derive it yourself from beginning to end.

The generators of Lorentz transformation should obey the following algebra:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\eta^{\mu\rho} J^{\nu\sigma} - i\eta^{\nu\rho} J^{\mu\sigma} + i\eta^{\mu\sigma} J^{\rho\nu} - i\eta^{\nu\sigma} J^{\rho\mu} \tag{1}$$

You can find a general proof in e.g. Weinberg's field theory textbook. Instead of asking you to prove it, I will ask you to check that this make sense in special cases.

a. Check that in $D = 4$, the above algebra reproduces the familiar commutator $[J^x, J^y] = iJ^z$, by making the identification $J^{12} = J^z$, $J^{23} = J^x$, $J^{31} = J^y$. Further, **check** that commuting 2 boosts result in a rotation by deriving $[K^x, K^y] = -iJ^z$ from the above algebra, and identifying the boosts in the three directions as $J^{10} = K^x$, $J^{20} = K^y$, $J^{30} = K^z$.

b. In the context of string theory, a natural guess for $J^{\mu\nu}$ is

$$J^{\mu\nu} = \int_0^\ell d\sigma (X^\mu(\sigma)\Pi^\nu(\sigma) - X^\nu(\sigma)\Pi^\mu(\sigma)) \tag{2}$$

which is analogous to $\vec{J} = \vec{x} \times \vec{p}$ that you learned in quantum mechanics. Note that we have suppressed τ in the arguments. Check that this choice makes sense by checking out the spatial rotations.

In other words, in the light cone gauge (which is what we will adopt hereafter), work out the commutator $[J^{ij}, J^{mn}]$, where i, j, m, n range over $2, 3, ..., D-1$, making use of the definition in eq. (2), and **show** that the resulting commutator agrees with eq. (1). You will find it useful to use the equal time commutator written down in class: $[X^i(\sigma), \Pi^j(\sigma')] = i\delta^{ij}\delta(\sigma - \sigma')$.

c. Next, we are interested $[J^{-i}, J^{-j}]$. First, **show** that the Lorentz algebra (eq. [1]) demands this to vanish. Be careful to use the correct metric for the light cone gauge: $\eta^{--} = \eta^{++} = \eta^{-i} = \eta^{+i} = 0$, etc.

d. What is non-trivial to check is that $[J^{-i}, J^{-j}]$ does vanish using the definition of J^{-i} according to eq. (2). As we will see, the vanishing of $[J^{-i}, J^{-j}]$ is by no means guaranteed for the quantum open string, and in fact appears to hold only for $D = 26$. According to eq. (2),

$$J^{-i} = \int_0^\ell d\sigma (X^- \Pi^i - X^i \Pi^-) \tag{3}$$

To write this out in terms of the string oscillators, we need expressions for X^- and Π^- which we have not worked out in class. The important thing to remember is that both are expressible in terms of X^i and its derivatives. Π^- can be obtained from $H = p^- = \int_0^\ell d\sigma \Pi^-$, which implies

$$\Pi^- = \frac{\ell}{4\pi\alpha' p^+}[2\pi\alpha' \Pi^{i^2} + \frac{1}{2\pi\alpha'}\partial_\sigma X^{i^2}] \tag{4}$$

Repeated i's are implicitly summed over, as usual.

To obtain an expression for $X^-(\tau, \sigma)$, one needs to work out the constraints from $T^\tau{}_\tau = 0 = T^\sigma{}_\tau$, which together imply

$$(\partial_\tau \pm c\partial_\sigma)X^- = \frac{1}{2}\sum_i (\partial_\tau X^i \pm c\partial_\sigma X^i)^2 \tag{5}$$

The above can be integrated to obtain

$$X^-(\tau, \sigma) = x^-(\tau = 0) + \frac{p^-}{p^+}\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^-}{n}e^{-\pi i n c\tau/\ell}\cos(\pi n\sigma/\ell) \tag{6}$$

which looks just like the analogous expression for the open string X^i, with the following definition:

$$\alpha_n^- \equiv \frac{1}{2p^+\sqrt{2\alpha'}}\sum_{\text{all}\, m}\alpha_{n-m}^i \alpha_m^i \tag{7}$$

where the summation is over all m's from $-\infty$ to ∞, including $m = 0$.

The above expressions for X^- and Π^-, together with the expressions for X^i and Π^i that we have written down in class, can be used to rewrite J^{-i} as

$$J^{-i} = x^-(\tau)p^i - x^i(\tau)p^- - \frac{i}{p^+\sqrt{2\alpha'}} \sum_{n>0} \frac{1}{n}(L_{-n}\alpha^i_n - \alpha^i_{-n}L_n) \tag{8}$$

where L_n are known as the Virasoro generators and are defined to be:

$$L_n \equiv \sqrt{2\alpha'}p^+\alpha^-_n = \frac{1}{2}\sum_{\text{all}\, m} \alpha^i_{n-m}\alpha^i_m \tag{9}$$

The above definition of L_n has no ordering ambiguity for $n \neq 0$. For $n = 0$, we define L_0 to be

$$L_0 = \frac{1}{2}{\alpha^i_0}^2 + \sum_{n>0} \alpha^i_{-n}\alpha^i_n \tag{10}$$

With this definition, the Hamiltonian is related to L_0 by

$$H = p^- = \frac{1}{2\alpha' p^+}[L_0 + A] \tag{11}$$

where A is an unknown ordering constant. You might recall that we worked out in class what A should be, using somewhat mysterious arguments concerning Casmir energy and the regularization of diverging sums. Here, we will instead take the point of view that the ordering of oscillators in H is ambiguous and so one should always include some kind of ordering constant in the definition of the Hamiltonian. We will see that demanding quantum Lorentz invariance fixes A to be exactly -1. This is more satisfactory in the sense that we don't have to think about regularizing diverging sums.

With the above definitions, we can rewrite eq. (8) as

$$J^{-i} = x^-p^i - \frac{x^i}{2\alpha' p^+}(L_0 + A) - \frac{i}{p^+\sqrt{2\alpha'}} \sum_{n>0} \frac{1}{n}(L_{-n}\alpha^i_n - \alpha^i_{-n}L_n) \tag{12}$$

Note that the small x's and p's are center of mass positions and momenta.

Note also that J^{-i} itself has its own ordering ambiguity. For instance x^i and L_0 do not commute. Neither do L_{-n} and α^i_n. It can actually be shown that alternative ordering introduces terms like p^i/p^+ into J^{-i} and such terms do not contribute to the commutator $[J^{-i}, J^{-j}]$, so we won't worry about them.

I won't ask you to derive eq. (12), but I will ask you to derive a few commutators that are useful in working out $[J^{-i}, J^{-j}]$.

Show that $[x^-, \frac{1}{p^+}] = \frac{i}{{p^+}^2}$.

Show that $[x^i, L_n] = i\sqrt{2\alpha'}\alpha^i_n$. You might find it useful to remember that $\alpha^i_0 = \sqrt{2\alpha'}p^i$.

Another commutator that is useful, but I won't ask you to derive is $[\alpha^i_n, L_m] = n\alpha^i_{n+m}$.

e. In addition to the above commutators, and the ones you already know, we will also need the important commutator of the Virasoro algebra:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{D-2}{12}(m^3 - m)\delta_{m+n,0} \tag{13}$$

Show that $[L_m, L_n] = (m-n)L_{m+n}$ for $m+n \neq 0$.

In your derivation above, you might be lead to believe that the result should hold even for $m+n=0$. An intermediate step that you probably encountered goes like this:

$$[L_m, L_n] = \frac{1}{2}(\sum_{\text{all}\, q} q\alpha^i_{m-q}\alpha^i_{n+q} + \sum_{\text{all}\, q}(m-q)\alpha^i_{m+n-q}\alpha^i_q) \tag{14}$$

You then probably redefined $q \to q-n$ in the first sum to obtain the desired result.

The point to keep in mind is that when $m+n=0$, both the first and second sums have their own ordering ambiguities which might result in infinite ordering constants. It can be a little dangerous to shift the q's when such ambiguities exist. Remember that our whole approach here is to avoid making assumptions about diverging sums. Let's instead allow for such ambiguities by writing in general

$$[L_m, L_n] = (m-n)L_{m+n} + B(m)\delta_{m+n,0} \tag{15}$$

We will see that $B(m)$ is non-zero. Note that by the above definition, $B(-m) = -B(m)$.

First, let us work out $B(1)$ and $B(2)$. It turns out $B(1)$ does vanish i.e. $[L_1, L_{-1}] = 2L_0$. Let us check $B(2)$:

$$\begin{aligned} [L_2, L_{-2}] &= \frac{1}{4}\sum_{\text{all}\, m,n}[\alpha^i_{2-m}\alpha^i_m, \alpha^j_{-2-n}\alpha^j_n] \qquad (16) \\ &= \frac{1}{4}[\alpha^i_1\alpha^i_1 + 2(\alpha^i_0\alpha^i_2 + \alpha^i_{-1}\alpha^i_3 + \alpha^i_{-2}\alpha^i_4 + ...), \alpha^j_{-1}\alpha^j_{-1} + 2(\alpha^i_{-2}\alpha^i_0 + \alpha^i_{-3}\alpha^i_1 + \alpha^i_{-4}\alpha^i_2 + ...)] \end{aligned}$$

The above commutator is of the form $[F + 2(G), F' + 2(G')]/4$. It can be seen that $[F, G'] = [G, F'] = 0$ because none of the pair of subscript indices sum to zero, so we can ignore these commutators.

Show that $[F, F']/4 = [\alpha^i_1\alpha^i_1, \alpha^j_{-1}\alpha^j_{-1}]/4 = (D-2)/2 + \alpha^i_{-1}\alpha^i_1$.

$[G, G']$ can also be worked out by noting that for instance $\alpha^i_0\alpha^i_2$ has non-trivial commutator only with $\alpha^i_{-2}\alpha^i_0$, and so on.

Show that $[\alpha^i_0\alpha^i_2, \alpha^i_{-2}\alpha^i_0] = 2 \times \alpha^i_0\alpha^i_0 - 0 \times \alpha^i_{-2}\alpha^i_2$

Show that $[\alpha^i_{-1}\alpha^i_3, \alpha^i_{-3}\alpha^i_1] = 3 \times \alpha^i_{-1}\alpha^i_1 - 1 \times \alpha^i_{-3}\alpha^i_3$.

By now, you can probably recognize the pattern: **Show** that $[G, G'] = 4L_0 - \alpha^i_{-1}\alpha^i_1$.

Finally, combine everything to **show** that $[L_2, L_{-2}] = 4L_0 + (D-2)/2$.

Therefore, we have shown that $B(2)$ does not vanish and is equal to $(D-2)/2$.

To obtain the general $B(m)$, we will use the Jacobi identity: $[L_k, [L_n, L_m]] + [L_n, [L_m, L, k]] + [L_m, [L_k, L_m]] = 0$. **Show** that this implies, for $n+m+k=0$, $(n-m)B(k) + (m-k)B(n) + (k-n)B(m) = 0$

Putting $k=1$ and $m = -n-1$ then gives

$$B(n+1) = (n-1)^{-1}[(n+2)B(n) - (2n+1)B(1)] \tag{17}$$

This gives a recurrence relation that allows us to determine $B(n)$ for arbitrary n given $B(2)$ and $B(1)$. The general solution is $B(m) = c_1 m + c_3 m^3$ as you can check by substitution.

Show that $B(1) = 0$ and $B(2) = (D - 2)/2$ tells us

$$B(m) = \frac{D-2}{12}(m^3 - m) \tag{18}$$

This completes the proof of eq. (13).

f. In case you have forgotten, we are interested in working out $[J^{-i}, J^{-j}]$! We now have all the commutators in place to carry this out. Unfortunately, the calculation is still rather lengthy. I will tell you most of the intermediate results, and will only ask you to compute a few selected quantities.

Let us define a quantity E^i:

$$E^i \equiv -\frac{i}{\sqrt{2\alpha'}} \sum_{n>0} \frac{1}{n}(L_{-n}\alpha^i_n - \alpha^i_{-n}L_n) \tag{19}$$

Eq. (12) can then be rewritten as

$$J^{-i} = x^- p^i - \frac{x^i}{2\alpha' p^+}(L_0 + A) + \frac{E^i}{p^+} \tag{20}$$

It turns out that

$$[x^- p^i - \frac{x^i}{2\alpha' p^+}(L_0 + A), x^- p^j - \frac{x^j}{2\alpha' p^+}(L_0 + A)] = 0 \tag{21}$$

and so we have

$$\begin{aligned} &[J^{-i}, J^{-j}] = [x^- p^i, E^j/p^+] + [E^i/p^+, x^- p^j] \\ &+[-\frac{x^i}{2\alpha' p^+}(L_0 + A), E^j/p^+] + [E^i/p^+, -\frac{x^j}{2\alpha' p^+}(L_0 + A)] \\ &+\frac{1}{{p^+}^2}[E^i, E^j] \end{aligned} \tag{22}$$

Let us work out some of these commutators.

Show that

$$[x^- p^i, E^j/p^+] = \frac{i}{{p^+}^2} p^i E^j \tag{23}$$

It can also be shown that

$$\begin{aligned} &[-\frac{x^i}{2\alpha' p^+}(L_0 + A), E^j/p^+] \\ &= \frac{1}{2\alpha' {p^+}^2}(-(L_0 + A)[x^i, E^j] + [E^j, L_0 + A]x^i) \\ &= \frac{-i}{2\alpha' {p^+}^2}(L_0 + A)E^{ij} \end{aligned} \tag{24}$$

where we have used $[E^j, L_0 + A] = 0$, and $[x^i, E^j] = iE^{ij} \equiv \sum_{n>0}(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n)/n$.

Therefore, we have

$$[J^{-i}, J^{-j}] = \frac{-i}{\alpha' {p^+}^2}(L_0 + A)E^{ij} + \frac{1}{{p^+}^2}[E^i, E^j] + \frac{i}{{p^+}^2}(p^i E^j - p^j E^i) \tag{25}$$

Believe it or not, even with all these manipulations, reducing the above commutator to a simple form is still rather formidable. We set ourselves the more modest goal of computing a certain expectation value of the above commutator, and checking the condition under which it vanishes. Let us sandwich $[J^{-i}, J^{-j}]$ between two states $< 0|\alpha^k_m$ and $\alpha^\ell_{-m}|0>$, with $m > 0$, and $|0>$ denoting the ground state of the string (the center of mass momentum eigenvalues are suppressed). Doing so for eq. (25), one obtains several terms

$$< 0|\alpha^k_m \frac{-i}{\alpha' p^{+2}}(L_0 + A)E^{ij}\alpha^\ell_{-m}|0 >= \frac{-m}{\alpha' p^{+2}}(\alpha' p^{q2} + A + m)(\delta^{ki}\delta^{j\ell} - \delta^{kj}\delta^{i\ell}) \tag{26}$$

Note that in the above, p^+ is used to represent both the operator and the eigenvalue (of state $|0>$).

$$\begin{aligned} &< 0|\alpha^k_m \frac{1}{p^{+2}}[E^i, E^j]\alpha^\ell_{-m}|0 > \\ &= \frac{-1}{2\alpha' p^{+2}}[(\delta^{ki}\delta^{j\ell} - \delta^{kj}\delta^{i\ell})(2m^2(m-1) + < 0|[L_{-m}, L_m]|0 >) \\ &+ 2\alpha' m(\delta^{j\ell}p^k p^i + \delta^{ki}p^j p^\ell - \delta^{i\ell}p^k p^j - \delta^{kj}p^i p^\ell)] \end{aligned} \tag{27}$$

$$< 0|\alpha^k_m \frac{i}{p^{+2}}(p^i E^j - p^j E^i)\alpha^\ell_{-m}|0 >= \frac{m}{p^{+2}}[\delta^{j\ell}p^i p^k - \delta^{kj}p^i p^\ell - \delta^{i\ell}p^j p^k + \delta^{ki}p^j p^\ell] \tag{28}$$

Combining the above three expressions, one obtains

$$\begin{aligned} &< 0|\alpha^k_m[J^{-i}, J^{-j}]\alpha^\ell_{-m}|0 > \\ &= \frac{-1}{2\alpha' p^{+2}}(\delta^{ki}\delta^{j\ell} - \delta^{kj}\delta^{i\ell})[2m(m + A) + 2m^2(m-1) + 2\alpha' m p^{q2} + < 0|[L_{-m}, L_m]|0 >] \end{aligned} \tag{29}$$

What remains to be done is to compute $< 0|[L_{-m}, L_m]|0 >$. **Show** that

$$< 0|[L_{-m}, L_m]|0 >= -2\alpha' m p^{q2} - \frac{(D-2)}{12}(m^3 - m) \tag{30}$$

where the index q is implicitly summed over as usual.

Finally, **show** that

$$< 0|\alpha^k_m[J^{-i}, J^{-j}]\alpha^\ell_{-m}|0 > \propto m[m^2(2 - \frac{D-2}{12}) + (2A + \frac{D-2}{12})] \tag{31}$$

and for the above to vanish for arbitrary $m > 0$, we need $D = 26$ and $A = -1$. In other words, unless these 2 conditions are met, $A = -1$ telling us the ordering constant in the Hamiltonian, and $D = 26$ telling us the spacetime dimension, quantum corrections violate Lorentz symmetry. The quantum violation of a classical symmetry is known as an anomaly.

Are you exhausted? Well, I am. But it is such an important result that it is good to know there is a way to derive it without dealing with dubious diverging sums. Later in the course, we will derive effectively the same result in other ways, but as far as I know, this tortuous calculation is the only way to show that the quantum string explicitly violates Lorentz symmetry unless $D = 26$. The arguments we went over in class concerning the counting of states only offer indirect evidence that Lorentz symmetry is violated, but had the advantage of taking much less work!