Introduction to String Theory G8050
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Problem Set 2, due 3/22/05

1. (*Optional.*) The Euclideanized Polyakov action with a flat world-sheet metric is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial^c X^\mu \partial_c X_\mu = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu) \tag{1}$$

This action is invariant under world-sheet translation where $\delta\sigma^a = \epsilon v^a$ and $\delta X^\mu = -\epsilon v^a \partial_a X^\mu$. Here v^a is just a constant vector, and ϵ is a small parameter. **Show** that the conserved current is given by

$$j^b = i v^a T_a{}^b \tag{2}$$

where $T_a{}^b$ is the world-sheet energy momentum tensor:

$$T_a{}^b = -\frac{1}{\alpha'}(\partial_a X^\mu \partial^b X_\mu - \frac{1}{2}\delta_a{}^b \partial^c X^\mu \partial_c X_\mu) \tag{3}$$

One way to prove this is to use this trick we discussed in class, where one considers a generalized transformation of the form $\delta X^\mu = -\epsilon\rho v^a \partial_a X^\mu$ where ρ is some arbitrary function of σ. Such a transformation would be equivalent to a world-sheet translation, and hence a symmetry of the action, only if ρ were a constant. Therefore, one expects the change in action under such a generalized transformation to be of the form

$$\delta S = -\frac{i\epsilon}{2\pi} \int d^2\sigma j^b \partial_b \rho \tag{4}$$

This provides a method to work out the conserved current j^b. **Show** that $\nabla_b j^b = 0$ using the equation of motion (Noether theorem already tells us that this current has to be conserved, but it is good to check that this is so explicitly using the equation of motion).

2. **Derive** the commutator

$$[\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{5}$$

using the XX OPE

$$X^\mu(z,\bar{z})X^\nu(z',\bar{z}') \sim -\frac{\alpha'}{2}\eta^{\mu\nu} \ln|z-z'|^2 \tag{6}$$

and the expression for α's in terms of a contour integral over ∂X:

$$\alpha^\mu_m = i\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \int \frac{dz}{2\pi i z} z^{m+1} \partial X^\mu(z) \tag{7}$$

Note that the above expressions, and hence your derivation, hold for both closed and open string.

(*Optional.*) **Derive** the commutator

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{8}$$

using similar arguments, for the cases of open string and closed string. Here, x^μ is the center of mass position of the string at $\tau = 0$ (or $\sigma^2 = 0$), and p^ν is the center of mass momentum. In other words, in the case of the closed string, x^μ and p^μ appear in the mode expansion as follows:

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m}\right) \tag{9}$$

The analogous expression for the open string is

$$X^\mu(z,\bar{z}) = x^\mu - i\alpha' p^\mu \ln|z|^2 + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{m\neq 0}\frac{\alpha_m^\mu}{m}\left(z^{-m} + \bar{z}^{-m}\right) \tag{10}$$

3. Normal ordering has to be done with some care for operators that sit at the boundaries of the world-sheet. Let's work it out here. Recall the definition of (conformal) normal ordering: we subtract the Green's function from $X^\mu(z,\bar{z})X^\nu(z',\bar{z}')$ so that it obeys the equation of motion. More explicitly, we know that (Polchinski eq. 2.1.20):

$$\frac{1}{\pi\alpha'}\partial_z\partial_{\bar{z}}X^\mu(z,\bar{z})X^\nu(z',\bar{z}') = -\eta^{\mu\nu}\delta^2(z-z',\bar{z}-\bar{z}') \tag{11}$$

We defined the normal order product to be

$$:X^\mu(z,\bar{z})X^\nu(z',\bar{z}') := X^\mu(z,\bar{z})X^\nu(z',\bar{z}') - G^{\mu\nu}(z-z',\bar{z}-\bar{z}') \tag{12}$$

where $G^{\mu\nu}(z-z',\bar{z}-\bar{z}')$ is the Green's function i.e. it satisfies

$$\frac{1}{\pi\alpha'}\partial_z\partial_{\bar{z}}G^{\mu\nu}(z-z',\bar{z}-\bar{z}') = -\eta^{\mu\nu}\delta^2(z-z',\bar{z}-\bar{z}') \tag{13}$$

The definition of normal ordering guarantees that $\partial_z\partial_{\bar{z}} :X^\mu(z,\bar{z})X^\nu(z',\bar{z}') := 0$ as an operator equation.

We have been adopting the following choice for the Green's function (we have been calling it a contraction):

$$G^{\mu\nu}(z-z',\bar{z}-\bar{z}') = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-z'|^2 \tag{14}$$

The above certainly does the job, but one might wonder whether other choices might make more sense. Here, 'other choices' means an alternative Green's function that obeys different boundary conditions. If one works in the $z = \exp[-i\sigma^1 + \sigma^2]$ coordinates, the above Green's function certainly satisfies the closed string boundary condition in the sense that it is periodic in σ^1 with a period of 2π. How about for the open string (where we like to use the coordinates: $z = -\exp[-i\sigma^1 + \sigma^2]$)? The open string boundary condition is equivalent to demanding that $\partial_z G^{\mu\nu}(z-z',\bar{z}-\bar{z}') = \partial_{\bar{z}}G^{\mu\nu}(z-z',\bar{z}-\bar{z}')$ for z on the real axis (this comes from the fact that $\partial_{\sigma^1}X^\mu$ vanishes at the boundary and so $\partial X^\mu = \bar{\partial}X^\mu$ on the real axis). **Show** that the following $G^{\mu\nu}$ satisfies the above open string boundary condition as well as, of course, eq. (13):

$$G^{\mu\nu}(z-z',\bar{z}-\bar{z}') = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-z'|^2 - \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-\bar{z}'|^2 \tag{15}$$

The extra (second) term on the right hand side can be thought of as originating from an image charge in the lower plane (assuming z' is in the upper plane). The reason why, so far, we have not cared about a (possible) second term is that if one confines oneself to z and z' in the upper plane and not on the real axis, the second term is never singular, and so we do not really care about it in the sense of an OPE (i.e. adding the second term only changes one's definition of normal ordering by a non-singular amount, and has no impact on the XX OPE). However, if z' is on the real axis, as z approaches z', one sees that both the first and second terms are singular – in fact the magnitude of the singularity is multiplied by a factor of 2. In other words, for two X's approaching each other on the boundary, one should subtract double the usual amount to create a non-singular normal ordered product (non-singular in the sense that the correctly normal ordered product should have an expectation value that is non-singular as z approaches z'). This is called boundary normal ordering.

4. Here, you are to work out the bc CFT. Recall that in class, we derived the following action for the ghosts:

$$S = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}}\, b_{ab} \hat{\nabla}^a c^b \tag{16}$$

where b_{ab} and c^b are Grassmann (i.e. anticommuting) fields, and $\hat{g}_{ab}$ is the gauge choice for the world sheet metric. Note that b_{ab} is symmetric traceless.

a. Let us adopt the flat world sheet metric, and $z, \bar{z}$ coordinates. **Show** that the above ghost action reduces to

$$S = \frac{1}{2\pi} \int d^2z (b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}}) \tag{17}$$

You will need to show, among other things, that $b_{z\bar{z}} = 0$.

Show that the above action is conformally invariant i.e. it is invariant under

$$\begin{aligned}
z' &= f(z) \\
b_{z'z'}(z') &= \left(\frac{\partial z'}{\partial z}\right)^{-2} b_{zz}(z) \\
c^{z'}(z') &= \frac{\partial z'}{\partial z} c^z(z) \\
b_{\bar{z}'\bar{z}'}(\bar{z}') &= \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-2} b_{\bar{z}\bar{z}}(\bar{z}) \\
c^{\bar{z}'}(\bar{z}') &= \frac{\partial \bar{z}'}{\partial \bar{z}} c^{\bar{z}}(\bar{z})
\end{aligned} \tag{18}$$

By the way, the above tells us that b_{zz}, c^z, $b_{\bar{z}\bar{z}}$ and $c^{\bar{z}}$ have weights $(2, 0), (-1, 0)$, $(0, 2), (0, -1)$ respectively. As a short hand, we can drop the subscripts and superscripts and write the action as

$$S = \frac{1}{2\pi} \int d^2z (b\bar{\partial}c + \tilde{b}\partial\tilde{c}) \tag{19}$$

Since the b, c pair and the $\tilde{b}, \tilde{c}$ pair are decoupled from each other, it is convenient to consider them separately i.e. let us first consider

$$S = \frac{1}{2\pi} \int d^2z b\bar{\partial}c \tag{20}$$

One should keep in mind, however, that eventually we would be interested in a theory with both b, c and $\tilde{b}, \tilde{c}$. Among other things, this is necessary for S to be real

– $\tilde{b}$ and $\tilde{c}$ are basically complex conjugates of b and c, as you can convince yourself by relating b_{zz}, c^z (and $b_{\bar{z}\bar{z}}$, $c^{\bar{z}}$) to b_{11} and b_{12} which are real.

(You might wonder what boundary conditions one should assume for the b, c fields. For closed string, it is pretty obvious: our world-sheet has no boundary, and so b, c should be appropriately periodic. For the open string, it is less obvious. We will show in class that in fact $b = \tilde{b}$ and $c = \tilde{c}$ on the real axis, and so one can perform the usual doubling trick to combine the b, c CFT and $\tilde{b}, \tilde{c}$ CFT into a single action as in eq. (20) such that the d^2z integral is over the whole complex plane. In other words, in computations below, you might find the need to integrate by parts, and you can ignore boundary terms in such cases.)

b. Once we arrive at the action in eq. (20), there is no reason not to be more general, and allow b to have conformal weights of $(\lambda, 0)$ (λ not necessarily equal to 2), and c to have weights $(1-\lambda, 0)$. **Show** that the above more general weights are consistent with the conformally invariant nature of S.

Derive the following operator equations:

$$\begin{aligned} \bar{\partial} c = 0 \quad , \quad \bar{\partial} b = 0 \\ \bar{\partial} b(z) c(z') = 2\pi \delta^2(z - z', \bar{z} - \bar{z}') \end{aligned} \tag{21}$$

where the first 2 equations of motion tell us c and b are holomorphic. The third equation motivates the following definition of normal ordering:

$$: b(z) c(z') := b(z) c(z') - \frac{1}{z - z'} \tag{22}$$

where $1/(z - z')$ is chosen such that the normal ordered product obeys the operator equation of motion: $\bar{\partial}_z : b(z)c(z') := 0$. In other words, **show** that

$$\partial_{\bar{z}} \frac{1}{z - z'} = \partial_z \frac{1}{\bar{z} - \bar{z}'} = 2\pi \delta^2(z - z', \bar{z} - \bar{z}') \tag{23}$$

The above may look a little strange, but you could find it useful to recall that $\partial \bar{\partial} \ln |z|^2 = 2\pi \delta^2(z, \bar{z})$.

The above tells us the OPE when b and c approach each other:

$$b(z) c(z') \sim \frac{1}{z - z'} \tag{24}$$

Since b and c are Grassmann fields, the above also implies

$$c(z) b(z') \sim \frac{1}{z - z'} \tag{25}$$

Finally, using similar arguments as above, **show** that $b(z)b(z') \sim 0$, and $c(z)c(z') \sim 0$.

c. Using similar arguments as in question 1, **show** that the energy momentum tensor for the bc theory (as specified by the action in eq. [20]) has the following components:

$$\begin{aligned} T(z) &\equiv T_{zz} = (\partial b) c - \lambda \partial (bc) \\ \tilde{T}(\bar{z}) &\equiv T_{\bar{z}\bar{z}} = 0 \\ T_{z\bar{z}} &= 0 \end{aligned} \tag{26}$$

One way to proceed is as follows: consider a transformation of the form $z' = z + \epsilon \rho(z, \bar{z}) v$, and $\bar{z}' = \bar{z} + \epsilon \rho(z, \bar{z})^* v^*$, which if $\rho(z, \bar{z})$ were equal to one, would be equivalent to a world-sheet translation (think of it as ${\sigma'}^a = \sigma^a + \epsilon v^a$, where $v^z = v$

and $v^{\bar z} = v^*$, with v and v^* being just constants). Under such a transformation, convince yourself that the fields should transform by $\delta b = -\lambda\epsilon(\partial\rho)vb - \epsilon\rho v\partial b - \epsilon\rho^* v^*\bar\partial b$ and $\delta c = -(1-\lambda)\epsilon(\partial\rho)vc - \epsilon\rho v\partial c - \epsilon\rho^* v^*\bar\partial c$. Note that one should not automatically set $\bar\partial b$ and $\bar\partial c$ to zero, though eventually, once you write down $\delta S = -(i\epsilon/2\pi)\int d^2z j_z\partial\rho + j_{\bar z}\bar\partial\rho$, you can think of everything as operator equations (in the sense of e.g. Polchinski eq. 2.3.5), and use the equation of motion to set $\bar\partial b = \bar\partial c = 0$. Finally, note that j_z and $j_{\bar z}$ are related to the energy momentum tensor by $j_z = iv^a T_{az}$ and $j_{\bar z} = iv^a T_{a\bar z}$.

We conformal normal order $T(z)$ as usual. **Derive** the central charges (c and $\tilde c$, not to be confused with the c of the bc fields) from the resulting TT OPE:

$$c = -3(2\lambda - 1)^2 + 1 \quad , \quad \tilde c = 0 \tag{27}$$

d. The appropriate Laurent expansions for the bc fields are

$$b(z) = \sum_m \frac{b_m}{z^{m+\lambda}} \quad , \quad c(z) = \sum_m \frac{c_m}{z^{m+1-\lambda}} \tag{28}$$

Use the bc OPE and contour integrals to **derive** the anticommutator:

$$\{b_m, c_n\} = \delta_{m+n,0} \tag{29}$$

It is not hard to see that $\{b_m, b_n\} = 0$ and $\{c_m, c_n\} = 0$.

Let us think about what the above commutator implies for the bc Fock space. Consider first the zero mode oscillators. They obey $\{b_0, c_0\} = 1$. This can be represented by a 2 state system $|0\rangle_0, |1\rangle_0$ ('spin' down and up), where $b_0|0\rangle_0 = 0$, $b_0|1\rangle_0 = |0\rangle_0$, $c_0|0\rangle_0 = |1\rangle_0$ and $c_0|1\rangle_0 = 0$. In other words, one can think of the two states as 'down': $|0\rangle_0$ and 'up': $b_0|0\rangle_0$. **Show** that this satisfies the $\{b_0, c_0\} = 1$ algebra, by representing b_0 as a matrix like: $\langle m|b_0|n\rangle$, and similarly for c_0. Let us consider oscillators associated with the next level of excitation. They obey the following algebra: $\{b_1, c_{-1}\} = 1, \{c_1, b_{-1}\} = 1$, and the rest of the anticommutators vanish. The story is very similar to the zero modes, except that here, the associated Fock space has 4 states: $|0\rangle_1$, $b_{-1}|0\rangle_1$, $c_{-1}|0\rangle_1$ and $b_{-1}c_{-1}|0\rangle_1$, with $|0\rangle_1$ satisfying $b_1|0\rangle_1 = 0$ and $c_1|0\rangle_1 = 0$. Note that the choice of b_{-1} as creation operators and b_1 as annihilation operators is somewhat arbitrary. One could have made the opposite choice. (Similarly, we have chosen b_0 to be the creation operator and c_0 to be the annihilation operator for the state $|0\rangle_0$.) The story is exactly the same for higher excitations. Therefore, one can think of the full Fock space this way. There is a full 'down' state which is a tensor product of the form $|0\rangle \equiv |0\rangle_0 \times |0\rangle_1 \times |0\rangle_2 \times |0\rangle_3$... and a full 'up' state which is $|1\rangle \equiv |1\rangle_0 \times |0\rangle_1 \times |0\rangle_2 \times |0\rangle_3$.... Both states are annihilated by b_n and c_n with $n > 0$. b_0 and c_0 serve to switch from one to another (or to annihilate them). The rest of the Fock space can be constructed by acting on $|0\rangle$ and $|1\rangle$ by the creation operators b_n and c_n with $n < 0$. Equations 2.7.18 of Polchinski can be understood this way.

e. Virasoro generators. Use the definition of Virasoro generators:

$$L_m = \int \frac{dz}{2\pi i z} z^{m+2} T(z) \tag{30}$$

and the mode expansions for b and c above to **show** that

$$L_m = \sum_n (m\lambda - n) : b_n c_{m-n} : + \delta_{m,0} a^g \tag{31}$$

Here : : denotes creation-annihilation normal ordering, and a^g is an ordering constant to be determined. **Show** that the ordering constant is $\lambda(1-\lambda)/2$ using whatever method you want. In particular, for the case of interest $\lambda = 2$, $a^g = -1$.

f. We have shown that the Hamiltonion in any CFT is given by

$$H = L_0 - \frac{c}{24} \tag{32}$$

where c is the central charge. (We have ignored the right moving part of the bc CFT, or you can think of the above as being appropriate for the open string).

Acting on the ground state $|0\rangle$ with the above, **show** that

$$H|0\rangle = a^g - \frac{c}{24} = \frac{1}{12} \tag{33}$$

This 1/12 can also be understood as Casmir energy as follows. Recall that for a scalar field, the Casmir energy is $-1/24$. Since the ghosts are fermions, the Casmir energy for each ghost should be 1/24, and since there are 2 ghost fields (b and c), the total comes to 1/12.

Finally, so far, we have ignored the right-moving sector: $\tilde{b}$ and $\tilde{c}$. The story is exactly analogous for these fields, in the case of the closed string. In the case of the open string, there is no need to think about them, by virtue of the doubling trick.

5. (*Optional.*) To make sure you understand how the whole approach from the path integral to commutators works, try working out a non-stringy example: consider a non-relativistic free point particle with action $S = \int dt \frac{1}{2}\dot{x}^2$, where t is time, $x(t)$ is the spatial position, and $\dot{x}$ its derivative. Using path integral techniques (Green's function, etc), **derive** the commutator $[x(t), p(t)] = i$, where p is the usual momentum conjugate to x.