G8050 Advanced Mathematical Methods: Introduction to String Theory
Instructor: Lam Hui
Problem Set 3, due 5/3/05

In this problem set, you are to work out the annulus amplitude with no external vertex operators. The result is given in eq. 7.4.1 of Polchinski without derivation. Here I will guide you through one possible derivation. A very similar approach is used by Polchinski to obtain the torus amplitude in Sections 7.2 and 7.3, which you might want to read. However, I will try to make this problem set as self-contained as possible – you should be able to tackle it based on nothing but your class notes.

In essence, we would like to compute $\langle 1 \rangle_{\rm annulus}$. (I say 'in essence', because what we will compute is almost the expectation value of unity, except for appropriate insertions of bc ghosts to take care of the modulus and CKV.) You might wonder: shouldn't the expectation value of unity be simply unity itself? That expectation would be correct if one defines expectation value of, say Q, to be $\int [dX] e^{-S} Q / \int [dX] e^{-S}$. (I use $[dX]$ here as a schematic measure that could include scalar fields, ghosts, etc.) However, we define the expectation value $\langle Q \rangle$ to be just $\int [dX] e^{-S} Q$, which is not automatically normalized to 1 for $Q = 1$. You might recall that in our calculations of tree amplitudes, there are these normalization factors floating around (which we called $C^X_{S_2}$, etc), which are related to determinants of differential operators such as the Laplacian. We could take this route and compute the relevant determinants for the annulus. This is slightly complicated. We will instead try a different tactic: use the more familiar operator method. But first, we need to derive the Fadeev-Popov determinant, this time taking into account both the CKVs *and* the moduli.

a. As discussed in class, the contribution to the scattering amplitude of n vertex operators from a diagram of a particular topology (e.g. sphere, disk, torus, annulus or something else) is

$$\int d^\mu t [dX] e^{-S[X,\hat g] - \lambda \chi_0} \Delta_{\rm FP}(\hat g, \hat\sigma) \left(\prod_{{\rm unfixed}\ a,i} \int d\sigma^a_i \right) \left(\prod_{i=1}^n \sqrt{\hat g} V_i(\sigma_i) \right) \tag{1}$$

Here, $S[X, \hat g]$ is the gauge-fixed Polyakov action, t represents each real modulus (there are μ of them), χ_0 is the Euler characteristic of the world sheet without vertex operators, σ^a_i indicates the world sheet coordinate of the ith vertex operator (V_i) in the a direction (i.e. a could be 1/2, or $z/\bar z$, etc). The Fadeev Popov determinant $\Delta_{\rm FP}$ is a function of the gauge choice for the world sheet metric $\hat g$ as well as those σ^a_i's that have been fixed (denoted collectively by $\hat\sigma$). There are κ fixed σ^a_i's, in accordance with κ real CKVs.

As discussed in class, it is important that the role of the moduli can be viewed in 2 different ways. One can think of the periodicities (of e.g. torus or annulus) as fixed, in which case different t's label different metrics (i.e. $\hat{g}$ is a function of t). One can alternatively think of the metric as fixed, in which case different t's label different periodicities (or shapes). In our derivation of the Fadeev Popov determinant here, the former is the more useful way of thinking about it. However, after we are done, when applying eq. (1) to actual problems, the latter interpretation is more convenient.

Eq. 1 was obtained in class by substituting the following into the un-gauge-fixed path integral and carrying out the usual manipulations:

$$1 = \int d^\mu t [d\zeta] \Delta_{\mathrm{FP}}[\hat{g}, \hat{\sigma}] \delta(g^\zeta - \hat{g}(t)) \prod_{\text{fixed } a,i} \delta(\sigma_i^{a\zeta} - \hat{\sigma}_i^a) \tag{2}$$

The above differs from from the last time we derived Δ_{FP} (for the CKVs) in that there is the additional integration of $d^\mu t$ here and that $\hat{g}$ now depends on t.

In writing down eq. (2), we have in mind that the delta function $\delta(g^\zeta - \hat{g}(t))$ picks out a particular ζ_0 and t_0 for which $g^{\zeta_0} - \hat{g}(t_0) = 0$. The integrals can then be rewritten as integration of δt and $\delta\zeta$ around t_0 and ζ_0 (similar things can be said for the fixed σ_i^a's, which basically partially enters into the definition of ζ_0). Therefore, we have

$$\Delta_{\mathrm{FP}}[\hat{g}, \hat{\sigma}]^{-1} = \int d^\mu \delta t [d\delta w d\delta\sigma] \delta(\delta g^{\delta w, \delta\sigma} - \delta g^{\delta t}) \prod_{\text{fixed } a,i} \delta(\delta\sigma^a(\hat{\sigma}_i)) \tag{3}$$

The delta function $\delta(\delta g^{\delta w, \delta\sigma} - \delta g^{\delta t})$ has two different contributions to its argument: one is the variation of the metric under diffeomorphism $\times$ Weyl transformations, and the other is the variation of the metric under variation of the moduli. As discussed in class, these two different variations of the metric are independent of each other (keeping periodicities fixed). The argument of this delta function can be written as

$$\delta g^{\delta w, \delta\sigma} - \delta g^{\delta t} \to \delta g_{ab} = (2\delta w - \nabla_c \delta\sigma^c)\hat{g}_{ab} - 2(\hat{P}_1 \delta\sigma)_{ab} - \delta t^k \partial_k \hat{g}_{ab} \tag{4}$$

where the first two terms on the right are the usual variations under diff. $\times$ Weyl and the last term is the variation under moduli (the derivative ∂_k is a derivative with respect to the modulus t^k – do not confuse this with derivative with respect to world sheet coordinates). The symbol $2(\hat{P}_1 \delta\sigma)_{ab}$ denotes $\nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a - (\nabla_c \delta\sigma^c)\hat{g}_{ab}$.

Eq. (3) can be rewritten using Fourier representations of delta functions:

$$\Delta_{\mathrm{FP}}^{-1} = \int d^\mu \delta t d^\kappa x [d\delta w d\delta\sigma d\beta] \exp\left[2\pi i \int d^2\sigma \sqrt{\hat{g}} \beta_{ab} \delta g^{ab} + 2\pi i \sum_{\text{fixed } a,i} x_{ai} \delta\sigma^a(\hat{\sigma}_i)\right] \tag{5}$$

where δg^{ab} is given in eq. (4). Note that the measure $[d\delta w d\delta\sigma d\beta]$ represents integration over these variables at every point on the world sheet. The integration over δw tells us that β_{ab} has to be tracefree. One can then relabel $\beta_{ab} \to \beta'_{ab}$, with the understanding that β' is traceless. Therefore, we have

$$\Delta_{\mathrm{FP}}^{-1} = \int d^\mu \delta t d^\kappa x [d\delta\sigma d\beta'] \tag{6}$$

$$\exp\left[-2\pi i\int d^2\sigma\sqrt{\hat{g}}\beta'_{ab}(2(\hat{P}_1\delta\sigma)^{ab}+\delta t^k\partial_k\hat{g}^{ab})+2\pi i\sum_{\text{fixed }a,i}x_{ai}\delta\sigma^a(\hat{\sigma}_i)\right]$$

To invert the Fadeev Popov determinant, we apply the standard trick of replacing everything by Grassmann variables:

$$\beta'_{ab}\to b_{ab}\quad,\quad \delta\sigma^a\to c^a\quad,\quad \delta t^k\to\xi^k\quad,\quad x_{ai}\to\eta_{ai} \tag{7}$$

We then have

$$\Delta_{\mathrm{FP}}(\hat{g},\hat{\sigma})=\int[dbdc]d^\mu\xi d^\kappa\eta \tag{8}$$

$$\exp\left[-\frac{1}{4\pi}\int d^2\sigma\sqrt{\hat{g}}b_{ab}(2(\hat{P}_1c)^{ab}-\xi^k\partial_k\hat{g}^{ab})+\sum_{\text{fixed }a,i}\eta_{ai}c^a(\hat{\sigma}_i)\right]$$

where we have been cavalier about signs and factors of 2π's, with the understanding that these can all be absorbed into definitions of the Grassmann variables and their integration measures.

The first part of the exponent $-\frac{1}{4\pi}\int d^2\sigma\sqrt{\hat{g}}b_{ab}2(\hat{P}_1c)^{ab}$ gives the familiar ghost action $-S_g$. **Show** that the integration over ξ and η gives

$$\Delta_{\mathrm{FP}}(\hat{g},\hat{\sigma})=\int[dbdc]e^{-S_g}\prod_{k=1}^{\mu}\frac{1}{4\pi}(b,\partial_k\hat{g})\prod_{\text{fixed }a,i}c^a(\hat{\sigma}_i) \tag{9}$$

where

$$(b,\partial_k\hat{g})\equiv\int d^2\sigma\sqrt{\hat{g}}b_{ab}\partial_k\hat{g}^{ab} \tag{10}$$

Putting the above into eq. (1), we finally arrive at the gauge-fixed path integral which fully takes into account CKVs and moduli:

$$\int\quad d^\mu t[dXdbdc]e^{-S_X-S_g-\lambda\chi_0}\prod_{k=1}^{\mu}\frac{1}{4\pi}(b,\partial_k\hat{g})\prod_{\text{fixed }a,i}^{\kappa}c^a(\hat{\sigma}_i) \tag{11}$$

$$\left(\prod_{\text{unfixed }a,i}\int d\sigma_i^a\right)\left(\prod_{i=1}^{n}\sqrt{\hat{g}}V_i(\sigma_i)\right)$$

The above path integral is complete other than possible discrete overcounting, which we won't worry about (see Polchinski Section 5.3).

This expression is also quite easy to remember: in the presence of CKVs, one fixes a suitable number of vertex operators and inserts c-ghosts at those fixed positions; in the presence of moduli, one inserts b-ghosts contracted with modulus-derivatives of the metric, and then integrate over the moduli.

From now on, we will adopt the (alternative) interpretation that in eq. (11), the metric $\hat{g}$ is fixed, and what the moduli t's control are the periodicities or shapes of the world sheet. (You might wonder what one should do about the modulus derivative $\partial_k\hat{g}$ under this perspective. One shouldn't set this to zero. Instead one should calculate this derivative adopting the perspective that $\hat{g}$ changes with the moduli, but evaluate this derivative at the fixed metric. This will become clearer when we work out the actual example of the annulus.)

(Incidentally, if you feel uncomfortable with all these rather formal manipulations that lead to the Fadeev Popov determinant, a good place to consult is Sydney Coleman's Aspects of Symmetry, p. 160 - 167.)

b. We want to apply eq. (11) to the annulus, which has one CKV ($\kappa = 1$) and one modulus ($\mu = 1$). The CKV is translation parallel to the boundary, and the modulus is the length of the annulus t. To make this explicit, suppose we use the Cartesian σ^1, σ^2 coordinates. The annulus spans the range $\sigma^1 = 0$ to $\sigma^1 = \pi$ and $\sigma^2 = 0$ to $\sigma^2 = 2\pi t$, with 0 and $2\pi t$ identified (i.e. $\sigma^2 \sim \sigma^2 + 2\pi t$). The CKV is translation in the σ^2 direction. In $w = \sigma^1 + i\sigma^2$ coordinates, on the other hand, we have $w \sim w + i2\pi t$, with Re w ranging from 0 to π. In the coordinates $z = -\exp[-iw]$, the annulus resides in the upper half plane between radius of unity and radius of $e^{2\pi t}$ with points at the two radii identified. It is sometimes useful to switch between these different coordinate descriptions, and we will stick to these notations to describe each.

Next, we do something that looks a bit illegal: we will apply eq. (11) as if we have external vertex operators on the annulus, but at the end we will set $n = 0$. This turns out to give the right answer. However, anticipating that we are interested in the case of no external vertex operators, let us not integrate out the delta function that fixes the position of the c-ghost insertion. In other words, our scattering amplitude (let us call it Z_{C_2}, C_2 for cylinder or annulus) is

$$Z_{C_2} = \int dt [dX dbdc] e^{-S_X - S_g} \frac{1}{4\pi} (b, \partial_t \hat{g}) \delta(\sigma^2 - \hat{\sigma}^2) c^2(\hat{\sigma}^1, \hat{\sigma}^2) \prod_{i=1}^{n} \mathcal{V}_i \tag{12}$$

Here, we have used the fact that $\chi_0 = 0$ for the annulus. Also, we have one CKV in the σ^2 direction, and therefore, we have a delta function fixing σ^2 to be at $\hat{\sigma}^2$, with a corresponding insertion of the component of the c-ghost in the σ^2 direction. The symbol $\mathcal{V}_i$ denotes the ith vertex operator, which has been appropriately integrated over the world sheet (either in the interior or along the boundaries). In other words, $\mathcal{V}_i = \int d^2\sigma_i \sqrt{g} V_i(\sigma_i)$ or $\mathcal{V}_i = \int ds_i V_i(\sigma_i)$ depending on whether it is a closed or open string state. Note how we have not used the delta function $\delta(\sigma^2 - \hat{\sigma}^2)$ to eliminate integration of any of the vertex operators. You might also wonder what $\hat{\sigma}^1$ is in the argument of c^2. You can imagine that one of our vertex operators live on the boundary, and so $\hat{\sigma}^1$ is at the boundary i.e. $\hat{\sigma}^1 = \pi$ or 0. As we will see below, in fact the choice of $\hat{\sigma}^1$ (or $\hat{\sigma}^2$ for that matter) has no impact on the final result.

From now on, we will forget about the external vertex operators altogether. We will also integrate over $\int d\sigma^2$ to get rid of the delta function, but then we need to divide by $\int d\sigma^2$ to get the right answer. Noting that $\int d\sigma^2 = 2\pi t$ for our annulus of length t, we arrive at

$$Z_{C_2} = \int \frac{dt}{2\pi t} [dX dbdc] e^{-S_X - S_g} \frac{1}{4\pi} (b, \partial_t \hat{g}) c^2(\hat{\sigma}^1, \hat{\sigma}^2) \tag{13}$$

You can think of the above expression intuitively as follows: to account for the overcounting due to CKVs, we divide by the volume of the Conformal Killing Group, which in this case happens to be $2\pi t$, where t also happens to be the modulus.

Our task is then neatly divided into evaluations of

$$Z^X_{C_2}(t) = \left[\int [dX] e^{-S_X}\right]_{\text{annulus t}} \quad , \tag{14}$$

$$Z^g_{C_2}(t) = \left[\int [dbdc] e^{-S_g} \frac{1}{4\pi}(b, \partial_t \hat{g}) c^2(\hat{\sigma}^1, \hat{\sigma}^2)\right]_{\text{annulus t}} \quad , \tag{15}$$

and

$$Z_{C_2} = \int_0^\infty \frac{dt}{2\pi t} Z^X_{C_2}(t) Z^g_{C_2}(t) \tag{16}$$

Let us first compute $Z^X_{C_2}(t)$. In operator language, such a path integral is equivalent to taking some state, propagate it along the annulus i.e. for a time $\tau = -2\pi i t$ (recall that $\tau = -i\sigma^2$ when we Euclideanize, and so $\sigma^2 = 2\pi t$ is equivalent to $\tau = -2\pi i t$), and then compute the resulting inner product with the original state itself, and finally summing over all possible states. In short:

$$Z^X_{C_2}(t) = \sum_s \langle s| e^{-iH\tau} |s\rangle = \sum_s \langle s| e^{-H 2\pi t} |s\rangle \tag{17}$$

Note that these states are open string states i.e. our string is stretched between $\sigma^1 = 0$ and $\sigma^1 = \pi$. You can think of the above as a thermal partition function if you'd like. Recall that $H = L_0 - \frac{c_X}{24}$ for the open string, we therefore have

$$Z^X_{C_2}(t) = q^{-c_X/24} \sum_s \langle s| q^{L_0} |s\rangle = q^{-c_X/24} \text{Tr.}\left[q^{L_0}\right] \quad , \quad q \equiv e^{-2\pi t} \tag{18}$$

Here, L_0 is the zero-mode Virasoro generator, which is related to the string oscillators by,

$$L_0 = \frac{1}{2}\alpha_0^\mu \alpha_{0\mu} + \sum_{n=1}^{\infty} \alpha^\mu_{-n}\alpha_{n\mu} = \alpha' p^2 + \sum_{n=1}^{\infty} \alpha^\mu_{-n}\alpha_{n\mu} \tag{19}$$

where the α's satisfy $[\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}$. A general state $|s\rangle$ takes the form

$$|s\rangle \propto ...[\alpha_{-3}]^{N_3}[\alpha_{-2}]^{N_2}[\alpha_{-1}]^{N_1}|0;k\rangle \tag{20}$$

where N_3, N_2, N_1, etc denote the number of excitations at the respective levels: they can each take a value anywhere from 0 to ∞. We have suppressed the μ space-time index: for each n, there are actually D α^μ_{-n}'s, corresponding to $\mu = 0, 1, 2, ..., D-1$. (We will keep D general for now, instead of setting it to 26 from the start.) In other words, strictly speaking, we should have written e.g. $[\alpha_{-1}]^{N_1}$ as $[\alpha^0_{-1}]^{N_1^0}[\alpha^1_{-1}]^{N_1^1}...[\alpha^{D-1}_{-1}]^{N_1^{D-1}}$ where

N_1^0, ... N_1^{D-1} can each independently take a value from 0 to ∞. The normalization constant is not exhibited explicitly, because we won't need it. It suffices to note our convention $\langle s|s\rangle = 1$. The center of mass momentum is k, which can be off the mass shell, since we are considering internal states.

Show using eq. (18), (19) and (20) that

$$Z^X_{C_2}(t) = q^{-c_X/24}\left[\sum_k q^{\alpha' k^2}\right]\left[\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right]^D \tag{21}$$

We treat k as if it takes discrete values: this would be the case if the space-time has a finite volume, say it fits into a box of size L on each side. The sum over k can be written as an integral: $\sum_k \to V_D \int d^D k/(2\pi)^D$, where V_D is the volume of our box. In the limit $V_D \to \infty$, our amplitude would diverge, but that is OK. What is physically relevant is the amplitude per unit space-time volume. Using the definition of q in eq. (18), we have

$$\sum_k q^{\alpha' k^2} = V_D \int \frac{d^D k}{(2\pi)^D} e^{-2\pi t\alpha' k^2} = V_D \int i\frac{d^D k_E}{(2\pi)^D} e^{-2\pi t \alpha' k_E^2} \tag{22}$$

where k_E represents the Euclideanized k i.e. $d^D k = dk^0 dk^1 ... dk^{D-1} = i dk^0_E dk^1_E .... dk^{D-1}_E$, with $k^0 = ik^0_E$, $k^1 = k^1_E$, ... $k^{D-1} = k^{D-1}_E$.

Show that

$$\sum_k q^{\alpha' k^2} = iV_D(8\pi^2 t\alpha')^{-D/2} \tag{23}$$

Putting everything together, we have the following scalar partition function on an annulus:

$$Z^X_{C_2}(t) = iV_D(8\pi^2 t\alpha')^{-D/2} q^{-c_X/24}\left[\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right]^D \tag{24}$$

The central charge c_X is of course equal to D because we have D scalar fields.

c. To calculate $Z^g_{C_2}(t)$ (eq. [15]), we need to first work out the modulus derivative term $(b, \partial_t \hat{g})/4\pi$. Using the $w, \bar{w}$ coordinates, this can be written as

$$\frac{1}{4\pi}(b, \partial_t \hat{g}) = \frac{1}{4\pi}\int d^2 w \sqrt{\hat{g}} b_{ab} \partial_t \hat{g}_{cd} \hat{g}^{ac} \hat{g}^{bd} = \frac{1}{2\pi}\int d^2 w b_{ww}(w,\bar{w}) \partial_t \hat{g}_{\bar{w}\bar{w}}(w,\bar{w}) \tag{25}$$

where we have chosen the canonical flat metric for $\hat{g}$ (i.e. $\hat{g}_{ww} = \hat{g}_{\bar{w}\bar{w}} = 0$ and $\hat{g}_{w\bar{w}} = 1/2$), , and we have taken advantage of the fact that we only have to worry about left-movers for the open string. Note that $b_{w\bar{w}} = 0$ by tracelessness of b.

The derivative $\partial_t \hat{g}_{\bar{w}\bar{w}}$ should be thought of as differentiating the modulus-dependent metric (that deviates in general from the canonical flat form), and then evaluating it at the canonical flat form. Let us make this explicit.

Consider an annulus with $ds^2 = dw' d\bar{w}'$, and $w' \sim w' + 2\pi i(t + \delta t)$: the metric is in the canonical flat form, but the periodicity is slightly off from $2\pi t$, the modulus of interest. Such an annulus can also be described by a *different* metric, keeping the periodicity fixed at $2\pi it$. The way to achieve this is to define a coordinate change $w' \to w$. $w = w' - (\delta t/2t)(w' - \bar{w}')$. With such a coordinate change, ignoring second order terms, one finds $w \sim w + 2\pi it$, and $ds^2 = (1 + \delta t/t)dwd\bar{w} - (\delta t/2t)dw^2 - (\delta t/2t)d\bar{w}^2$. The metric now deviates slightly from the canonical flat form. Differentiating gives $\partial_t \hat{g}_{\bar{w}\bar{w}} = -1/(2t)$.

The modulus derivative term is therefore

$$\frac{1}{4\pi}(b, \partial_t \hat{g}) = \frac{-1}{4\pi t}\int d^2 w b_{ww}(w, \bar{w}) \tag{26}$$

Let us put this into eq. (15), and make use of the following expansions of b and c in terms of the ghost oscillators:

$$b_{ww}(w, \bar{w}) = -z^2 b_{zz}(z, \bar{z}) = -\sum_n \frac{b_n}{z^{n+2}} z^2 = -\sum_n b_n(-e^{-iw})^{-n} \tag{27}$$
$$c^2(\hat{\sigma}^1, \hat{\sigma}^2) = \frac{1}{2}\sum_m c_m[(-e^{-i\hat{w}})^{-m} + (-e^{i\hat{\bar{w}}})^{-m}]$$

where $\hat{w} = \hat{\sigma}^1 + i\hat{\sigma}^2$ and $\hat{z} = e^{-i\hat{w}}$, and we have used the fact that $c_m = \tilde{c}_m$ for open string ghosts. We therefore have the following ghost partition function:

$$Z^g_{C_2}(t) = \frac{1}{4\pi t}\int d^2 w \sum_{n,m}(-e^{-iw})^{-n}\frac{1}{2}[(-e^{-i\hat{w}})^{-m} + (-e^{i\hat{\bar{w}}})^{-m}]\langle b_n c_m\rangle \tag{28}$$

with

$$\langle b_n c_m\rangle = \int [dbdc] e^{-S_g} b_n c_m \tag{29}$$

Just like what we did for the X path integral, we can express $\langle b_n c_m\rangle$ on the annulus in operator language as

$$\langle b_n c_m\rangle = \sum_s \langle s|b_n c_m e^{-H 2\pi t}(-1)^F|s\rangle \tag{30}$$

where $|s\rangle$ represents the ghost states. There are two differences from the analogous expression in eq. (17): here, there are insertions b_n and c_m, and there is an extra factor of $(-1)^F$, which assigns sign to a state based on whether there is an even ($(-1)^F = 1$) or odd ($(-1)^F = -1$) number of creations operators associated with that state. This factor arises as a result of Grassmannian nature of the path integral. To derive this will be too much of a digression from the task at hand. You can consult Polchinski's Appendix A.2 or the discussions at the end of this problem set if you would like to see a proof.

Taking eq. (30) as a given, let us rewrite:

$$\langle b_n c_m\rangle = q^{-c_g/24}\sum_s \langle s|b_n c_m q^{L_0}(-1)^F|s\rangle \quad , \quad q \equiv e^{-2\pi t} \tag{31}$$

where c_g is the ghost central charge, and

$$L_0 = \sum_{n=\infty}^{\infty}(-n) : b_n c_{-n} : -1 \tag{32}$$
$$= (b_{-1}c_1 + 2b_{-2}c_2 + 3b_{-3}c_3 + ...) + (c_{-1}b_1 + 2c_{-2}b_2 + 3c_{-3}b_3 + ...) - 1$$

where :: denotes creation-annihilation normal ordering, and note that $\{b_m, c_n\} = \delta_{m+n,0}$. For the zero modes, we identify b_0 as annihilation and c_0 as creation.

Recall from problem set 2 that the whole ghost Hilbert space can be built from two states $|0\rangle$ and $|1\rangle$, where $b_0|0\rangle = 0$, and $c_0|0\rangle = |1\rangle$ (and $c_0|1\rangle = 0$, $b_0|1\rangle = |0\rangle$). Since b_0 is regarded as annihilation and c_0 as creation, $|0\rangle$ has $(-1)^F = 1$, and $|1\rangle$ has $(-1)^F = -1$. The rest of the states can be obtained by acting on them with creation operators i.e. the Hilbert space has two branches: one consists of $|0\rangle$, $b_{-1}|0\rangle$, $c_{-1}|0\rangle$, $b_{-2}|0\rangle$, $c_{-2}|0\rangle$, $b_{-1}c_{-1}|0\rangle$, ... while the other branch consists of $|1\rangle$, $b_{-1}|1\rangle$, $c_{-1}|1\rangle$, $b_{-2}|1\rangle$, $c_{-2}|1\rangle$, $b_{-1}c_{-1}|1\rangle$, ... There is a one-to-one correspondence between these two branches, with the corresponding member of the opposite branch always carrying the opposite $(-1)^F$. Note that $b_n|0\rangle = b_n|1\rangle = c_n|0\rangle = c_n|1\rangle = 0$ for $n > 0$.

Let's try operating q^{L_0} on some of these states just to get a sense of how things work: $q^{L_0}|0\rangle = q^{-1}|0\rangle$, $q^{L_0}b_{-1}|0\rangle = q^0 b_{-1}|0\rangle$, $q^{L_0}b_{-1}c_{-1}|0\rangle = q^1 b_{-1}c_{-1}|0\rangle$, $q^{L_0}|1\rangle = q^{-1}|1\rangle$, $q^{L_0}b_{-1}|1\rangle = q^0 b_{-1}|1\rangle$, $q^{L_0}b_{-1}c_{-1}|1\rangle = q^1 b_{-1}c_{-1}|1\rangle$, etc. Note how q^{L_0} does not change the sign $(-1)^F$ of the states i.e. $|s\rangle$ and $q^{L_0}|s\rangle$ have the same $(-1)^F$.

As a warm-up exercise, let us compute $\langle 1\rangle$ on an annulus for the ghosts i.e. replace $b_n c_m$ in eq. (31) by unity:

$$\langle 1\rangle = q^{-c_g/24} \sum_s \langle s|q^{L_0}(-1)^F|s\rangle \tag{33}$$

Show that the above expectation value vanishes. You might find it useful to remember that there are these two branches of the ghost state space with opposite $(-1)^F$. This is why the ghost insertions corresponding to the modulus and CKV are so important: without them, the path integral would simply vanish.

Next, **show** that eq. (31) gives a vanishing result, except for $n = 0$ and $m = 0$, and **show** that

$$\langle b_n c_m\rangle = \delta_{n,0}\delta_{m,0} q^{-c_g/24} q^{-1} \left[\prod_{j=1}^{\infty}(1-q^j)\right]^2 \tag{34}$$

Putting the above into eq. (28), we obtain

$$Z^g_{C_2}(t) = \pi q^{-c_g/24} q^{-1} \left[\prod_{j=1}^{\infty}(1-q^j)\right]^2 \tag{35}$$

where we have used $\int d^2w = 4\pi^2 t$ (think of it like this: $\int d\sigma^1 d\sigma^2 = \int d^2w/2 = \pi \times 2\pi t$). The fact that only the $n = 0, m = 0$ modes contribute to eq. (28) proves our earlier assertion that the result is independent of the choice of $\hat{w}, \hat{\bar{w}}$.

Finally, putting everything together: **show** that eq. (24), (35) together with eq. (16) imply

$$Z_{C_2} = \int_0^\infty \frac{dt}{2t} \left(iV_D(8\pi^2 t\alpha')^{-D/2}\right) e^{-(c_X+c_g)/24} q^{-1} \left[\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right]^{D-2} \tag{36}$$

$$= \int_0^\infty \frac{dt}{2t} \left(iV_{26} (8\pi^2 t\alpha')^{-13} \right) \eta(it)^{-24}$$

where we have put $D = 26$ for the last equality, and $\eta(it) \equiv e^{-\pi t/12} \prod_{n=1}^{\infty} (1 - e^{-2n\pi t})$. The above expression agrees with Polchinski's eq. 7.4.1 aside from the Chan-Paton factor of n^2 which we have ignored. (We will develop the idea of Chan-Paton factors via D-branes.)

This completes all the derivations I want you to do for this problem set. There are several remarks to be made about this expression. If you are interested, read on.

- This annulus amplitude, with small modifications, is related to the tension of D-branes. We hope to develop this connection in class. See discussion in Polchinski section 8.7.

- Let us study where each of the terms in the first line of eq. (36) come from. The factor of $(iV_D(8\pi^2 t\alpha')^{-D/2})$ comes from the center-of-mass modes of the X's. The factor of $[\prod_{n=1}^{\infty} 1/(1-q^n)]^{D-2}$ comes from the internal excitations of the X's: there are D directions, but 2 of them lead to unphysical modes; ghosts precisely remove these 2 unphysical degrees of freedom. The factor of q^{-1} comes from the ghosts. One could therefore rewrite the complete partition function in a particularly simple way:

$$Z_{C_2} = \int_0^\infty \frac{dt}{2t} \,\mathrm{Tr.}'[e^{-2\pi t(L_0 - 1)}] \tag{37}$$

  where Tr.$'$ denotes trace over all the physical states (which can be off mass shell) associated with the X's. The factor of -1 is related to the normal ordering constant that we talked about way back (c.f. light cone quantization). The integration over t corresponds to counting all possible shapes (moduli) of the annulus. The division by $2t$ arises from dividing out the volume of the Conformal Killing Group.

- Even ignoring V_{26}, eq. (36) is divergent due to the small t contributions to the integral. It turns out this divergence can be reinterpreted as an IR divergence. See Polchinski section 7.4 for more discussions.

- You might wonder whether it is possible to compute the annulus amplitude as if it were a cylinder: i.e. a closed string propagating from one end to another. This is indeed possible. One has to carefully define the states associated with the ends of the cylinder. See Polchinski Section 7.4.

- You might wonder whether we should have divided eq. (36) by a factor of 2 to account for discrete overcounting: $\sigma^a \to -\sigma^a$ also leaves the metric (and periodicities) unchanged, rather like the CKVs. (Just flipping sign on one of the σ's is relevant only for unoriented string.) This is the factor of n_R in Polchinski's eq. 5.3.9. Indeed, we should have divided by 2, but then, we were not exactly careful in setting $\int d^2w = 4\pi^2 t$ in eq. (35): in eq. (26) for instance, we have used the doubling trick to eliminate the need for $b_{\bar{w}\bar{w}}$, and so the integration over w should actually be twice as large. Therefore, had we been very careful, the factors of 2 cancel out, so our final result remains valid.

- To make this problem set as self-contained as possible, let us derive the mysterious factor of $(-1)^F$. Consider a two-state system: $|0\rangle, |1\rangle$ orthonormal, with the operators a and $a^\dagger$ satisfying $a|0\rangle = 0$, $a^\dagger|0\rangle = |1\rangle$, $a|1\rangle = |0\rangle$, $a^\dagger|1\rangle = 0$ and $\{a, a^\dagger\} = 1$. (a and $a^\dagger$ are analogs of b_0 and c_0; we will ignore the non-zero modes for simplicity.) A coherent state $|\zeta\rangle$ is defined to be $e^{a^\dagger\zeta}|0\rangle = |0\rangle + |1\rangle\zeta$, where ζ is a Grassmann variable. Note that $a|\zeta\rangle = |\zeta\rangle\zeta$. Let us define the adjoint $\langle\zeta| = -\langle 1| + \zeta\langle 0|$ such that $\langle\zeta|\zeta'\rangle = \zeta - \zeta'$. This makes sense because $\zeta - \zeta'$ functions as a delta function in Grassmann integrals: $\int d\zeta(\zeta - \zeta')f(\zeta) = f(\zeta')$. Also $\langle\zeta|a = -\zeta\langle\zeta|$. It can be shown that

$$\int |\zeta\rangle d\zeta\langle\zeta| = |0\rangle\langle 0| + |1\rangle\langle 1| = 1 \tag{38}$$

Now, consider

$$\text{Tr.}(-1)^F A = \sum_n \langle n|(-1)^F A|n\rangle = \int \sum_n \langle n|(-1)^F|\zeta\rangle d\zeta\langle\zeta|A|n\rangle \tag{39}$$

Consider $\langle 0|(-1)^F|\zeta\rangle = 1 = \langle 0|\zeta\rangle$ and $\langle 1|(-1)^F|\zeta\rangle = -\zeta = -\langle 1|\zeta\rangle$. The moral is that $\langle n|(-1)^F|\zeta\rangle = \langle n|\zeta\rangle$ if $\langle n|\zeta\rangle$ is a c-number, while $\langle n|(-1)^F|\zeta\rangle = -\langle n|\zeta\rangle$ if $\langle n|\zeta\rangle$ is Grassmann. Therefore, $\langle n|(-1)^F|\zeta\rangle d\zeta = d\zeta\langle n|\zeta\rangle$. Finally, the order of $\langle n|\zeta\rangle$ and $\langle\zeta|A|n\rangle$ can be interchanged, because the only non-zero contribution to the integral comes from terms where $\langle n|\zeta\rangle$ is c-number and $\langle\zeta|A|n\rangle$ is Grassmann, or vice versa. Therefore, we have

$$\text{Tr.}(-1)^F A = \int d\zeta\langle\zeta|A|\zeta\rangle \tag{40}$$

The path integral on an annulus corresponds to the above with $A = e^{-H2\pi t}$.