Physics G8099/V3400: String Theory
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Here is a collection of useful expressions.

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-h} h^{ab} \partial_a X \partial_b X \tag{1}$$

where σ ranges from 0 to ℓ.

The diff. $\times$ Weyl invariance can be used to set $h_{ab} = \eta_{ab}$ in local patches.

Euclideanized version: $iS = -S_E$ is

$$S_E = \frac{1}{4\pi\alpha'} \int d^2\sigma \delta^{ab} \partial_a X \partial_b X \tag{2}$$

where $\sigma = \sigma^1$ and $i\tau = \sigma^2$. Hence, $\sigma - \tau = \sigma^1 + i\sigma^2$, and $\sigma + \tau = \sigma^1 - i\sigma^2$.

It is further useful to consider either $z = \sigma^1 + i\sigma^2$, or $z = e^{-i(\sigma^1 + i\sigma_2)(2\pi/\ell)}$ [closed], $z = e^{-i(\sigma^1 + i\sigma_2)(\pi/\ell)}$ [open]. We will mostly use the latter definition(s) of z. In all cases, we have

$$S_E = \frac{1}{2\pi\alpha'} \int dz d\bar{z} \partial X \bar{\partial} X \tag{3}$$

With the above definitions, it is natural to choose $\ell = 2\pi$ for closed string and $\ell = \pi$ for open string, although one doesn't have to.

Mode expansions.

For closed string:

$$X^\mu = x^\mu + \alpha' \frac{2\pi}{\ell} p^\mu \tau + i \left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n \neq 0} \left[\frac{1}{n} \alpha_n^\mu e^{in(\sigma - \tau)2\pi/\ell} + \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in(\sigma+\tau)2\pi/\ell} \right] \tag{4}$$
$$= x^\mu - i\frac{\alpha'}{2} p^\mu \ln|z|^2 + i \left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n \neq 0} \left[\frac{\alpha_n^\mu}{z^n} + \frac{\tilde{\alpha}_n^\mu}{\bar{z}^n} \right]$$

For open string:

$$X^\mu = x^\mu + \alpha' \frac{2\pi}{\ell} p^\mu \tau + i \left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n \neq 0} \left[\frac{1}{n} \alpha_n^\mu e^{in(\sigma - \tau)\pi/\ell} + \frac{1}{n} \alpha_n^\mu e^{-in(\sigma+\tau)\pi/\ell} \right] \tag{5}$$
$$= x^\mu - i\alpha' p^\mu \ln|z|^2 + i \left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n \neq 0} \left[\frac{\alpha_n^\mu}{z^n} + \frac{\tilde{\alpha}_n^\mu}{\bar{z}^n} \right]$$

In both cases, one can write

$$\partial X^\mu = -i \left(\frac{\alpha'}{2}\right)^{1/2} \sum_n \frac{\alpha_n^\mu}{z^{n+1}} \tag{6}$$
$$\bar{\partial} X^\mu = -i \left(\frac{\alpha'}{2}\right)^{1/2} \sum_n \frac{\tilde{\alpha}_n^\mu}{\bar{z}^{n+1}}$$

except that for open string, $\tilde{\alpha}^\mu_n = \alpha^\mu_n$, and that $p^\mu = (2/\alpha')^{1/2}\alpha^\mu_0 = (2/\alpha')^{1/2}\tilde{\alpha}^\mu_0$ [closed], $p^\mu = (1/2\alpha')^{1/2}\alpha^\mu_0$ [open].

The inverse for both open and closed string is:

$$\alpha^\mu_n = \left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{dz}{2\pi} z^n \partial X^\mu \tag{7}$$

and similarly for the anti-holomorphic part for closed string:

$$\tilde{\alpha}^\mu_n = -\left(\frac{2}{\alpha'}\right)^{1/2} \oint \frac{d\bar{z}}{2\pi} \bar{z}^n \bar{\partial} X^\mu \tag{8}$$

Here, the integral over z or $\bar{z}$ should be understood as a contour integral. For the open string, since we choose $\ell = \pi$, the contour integral should strictly speaking extend only over half the complex plane. We extend the contour to cover the full circle by a doubling trick: simply define ∂X^μ beyond the original half plane such that the expression for α^μ_n makes sense for a full circle.

Upon quantization, one finds

$$[\alpha^\mu{}_m, \alpha^\nu{}_n] = \eta^{\mu\nu} m \delta_{m,-n} \tag{9}$$

which follows from the Green function

$$\langle X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2)\rangle = -\frac{1}{\alpha'}\eta^{\mu\nu} \ln |z_1 - z_2|^2 \tag{10}$$

Note that because X is real, $\alpha_{-m} = \alpha_m{}^\dagger$. It is natural to interpret $\alpha_{-m}/\sqrt{m}$ for postive m as an operator that create internal oscillations.

Virasoro algebra.

The classical equation of motion from varying h_{ab} tells us that

$$T_{ab} = 0 \quad , \quad T_{ab} = -\frac{1}{\alpha'}[\partial_a X \partial_b X - \frac{1}{2} h_{ab} \partial^c X \partial_c X] \tag{11}$$

In the quantum theory, this turns into a constraint on the physical spectrum, roughly speaking, we want $T_{ab}|\phi\rangle = 0$ for physical state $|\phi\rangle$.

In conformal gauge, with the $z, \bar{z}$ coordinates, we have

$$T(z) \equiv T_{zz}(z) = -\frac{1}{\alpha'} : \partial X \partial X : \quad , \quad \tilde{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X \bar{\partial} X : \tag{12}$$

where the colons denote normal ordering, something we need to specify if one thinks of T and $\tilde{T}$ as quantum operators. Here, we define normal ordering for any operator $\mathcal{O}$ to mean

$$: \mathcal{O} :\equiv \mathcal{O} - \langle \mathcal{O} \rangle \tag{13}$$

We can define a mode expansion for T and $\tilde{T}$, just like for ∂X and $\bar{\partial} X$:

$$T(z) = \sum_m \frac{L_m}{z^{m+2}} \quad , \quad \tilde{T}(\bar{z}) = \sum_m \frac{\tilde{L}_m}{\bar{z}^{m+2}} \tag{14}$$
$$L_m = \oint \frac{dz}{2\pi i} z^{m+1} T(z) \quad , \quad \tilde{L}_m = -\oint \frac{d\bar{z}}{2\pi i} \bar{z}^{m+1} \tilde{T}(\bar{z})$$

One can derive

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \tag{15}$$

where c is known as the central charge, and for the X CFT, $c = D =$ number of space-time dimensions. An analogous expression holds for $\tilde{L}_m$.

The above expressions hold for both open and closed string, but the anti-holomorphic parts are redundant for open string.

One can express the L_m's in terms of the α_m^μ's:

$$L_m = \frac{1}{2} \sum_n : \alpha_n \alpha_{m-n} : \tag{16}$$

where the sum over μ index is suppressed. Here, the colons refer to normal ordering in the more familiar sense: creation (α_n with negative n) to the left and annihilation (α_n with positive n) to the right. It can be shown that the previous normal ordering and this normal ordering are equivalent, though this is a subtle thing because it is coordinate dependent (see Polchinski chapter 2 for details). The only operator for which one needs worry about normal ordering at all is L_0. One can easily check that Eq. (16) for L_0 with no additional normal ordering constant is correct, by using Eq. (15): $L_0 = \frac{1}{2}[L_1, L_{-1}]$. Letting this act on the vacuum state (i.e. the state annihilated by α_n for all $n \geq 0$), one can see that there can't be any constant in addition to Eq. (16) in the case of L_0.

Let's write out explicit expressions for L_0, L_1 and L_{-1}:

$$L_0 = \frac{1}{2}\alpha_0\alpha_0 + \sum_{n>0} \alpha_{-n}\alpha_n \tag{17}$$
$$L_1 = \frac{1}{2}(\alpha_0\alpha_1 + \alpha_{-1}\alpha_2 + ...)$$
$$L_{-1} = \frac{1}{2}(\alpha_{-1}\alpha_0 + \alpha_{-2}\alpha_1 + ...)$$

Antiholomorphic counterparts exist for the closed string.

Spectrum.

The commutator Eq. (9) tells us that the spectrum should be something like $|k\rangle$, $\alpha_{-1}^\mu|k\rangle$ and so on, where $|k\rangle$ is a state of no internal oscillations and center of mass momentum k^μ. We know this can't be the whole story because some of these states (namely the states with $\mu = 0$) have negative norm.

What we need to do is to impose the physical state condition and mod out the null states. The physical state condition is

$$L_n + A\delta_{n,0}|\phi\rangle = 0 \quad n \geq 0 \tag{18}$$

We allow for a constant A because so far, we have *defined* T or the L_m's such that Eq. (16) holds, but there is no compelling reason for choosing this definition over other normal ordering prescriptions which would have added constant A to L_0. As we have argued in class, A should be chosen to be -1. $A < -1$ can be ruled out because it allows negative norm states, $A > -1$ looks at first glance OK but turns out not to give consistent interactions. If you like, a rigorous justification is via light cone quantization and in problem set, you have shown that $A = -1$ and $D = 26$ is the only way to preserve Lorentz invariance in the quantum theory. In covariant quantization, a rigorous justification of the same can be obtained by considering the effects of ghosts and demanding that the theory has no conformal anomaly (i.e. total central charge of zero), in which case $A = -1$ arises from L_0 for the ghosts.

What does moding out null states mean? First, we need to define spurious states:

$$|\chi\rangle = \sum_{n>0} L_{-n}|\chi_n\rangle \tag{19}$$

where $|\chi_n\rangle$ is any state. Such a state χ is orthogonal to all physical states.

A null state is a state that is both physical and spurious. Such a state necessarily has zero norm.

The true Hilbert space should be defined by equivalence class:

$$|\phi\rangle \sim |\phi\rangle + |\,\mathrm{null}\rangle \tag{20}$$

i.e. physical states that are related by additions of null states should be considered equivalent. This is what we mean by moding out null states. This is very reasonable, because $|\phi\rangle$ and $|\phi\rangle + |\,\mathrm{null}\rangle$ give exactly the same expectation values when dotted with any physical states.

For closed string, one supplements Eqs. (18) and (19) by there antiholomorphic counterparts. In particular, the same $A = -1$ holds for $\tilde{L}_0$. One implication is therefore that

$$(L_n - \tilde{L}_n)|\phi\rangle 0 \quad n \geq 0 \tag{21}$$

This is sometimes called level matching.

Setting $A = -1$, the open string spectrum is as follows:

$$|k\rangle \quad , \quad e_\mu \alpha^\mu{}_{-1}|k\rangle \quad , \quad ... \tag{22}$$

The ground state (ground in the sense of no internal excitations) $|k\rangle$ has mass of $m^2 = -1/\alpha'$. We will call it the tachyon. The first excited state $e_\mu \alpha^\mu{}_{-1}|k\rangle$ has a mass of $m^2 = 0$ (which means $k^2 = -m^2 = 0$), and $e_\mu k^\mu = 0$ with the equivalence class $e_\mu \sim e_\mu + \lambda k_\mu$ where λ is an arbitrary constant. This means there are only $D-2$ distinct polarizations associated with e_μ's. This is a massless vector particle which contains only transverse polarizations and we will call it the 'photon'.

The closed string spectrum goes like:

$$|k\rangle \quad , \quad e_{\mu\nu} \alpha^\mu{}_{-1} \tilde{\alpha}^\nu_{-1}|k\rangle \quad , \quad ... \tag{23}$$

The ground state (ground in the sense of no internal excitations) $|k\rangle$ has mass of $m^2 = -4/\alpha'$. The first excited state $e_{\mu\nu} \alpha^\mu{}_{-1} \tilde{\alpha}^\nu_{-1}|k\rangle$ has a mass of $m^2 = 0$ (which means $k^2 = -m^2 = 0$). The conditions on $e_{\mu\nu}$ are:

$$k^\mu e_{\mu\nu} = 0 \quad , \quad k^\nu e_{\mu\nu} = 0 \quad , \quad e_{\mu\nu} \sim e_{\mu\nu} + k_\mu b_\nu + a_\mu k_\nu \tag{24}$$

where b_ν and a_μ are arbitrary constant vectors. The above conditions, together with the fact that $k^2 = 0$, tells us that the polarization tensor $e_{\mu\nu}$ contains only $(D-2)^2$ distinct true degrees of freedom. More concretely, suppose we go to a frame in which $k^\mu = (E, E, 0, 0, ..., 0)$ where there are $D-2$ zeros. Then, the true degrees of freedom of $e_{\mu\nu}$ are contained in the submatrix: e_{ij} where $i, j = 2, 3, ...D-1$. Here, $e_{\mu\nu}$ (or e_{ij}) can always be decomposed into a symmetric traceless part (graviton), a trace (dilaton) and an antisymmetric part (axion).

One might wonder why the closed spectrum does not contain something like the photon e.g. $\alpha^\mu{}_{-1}|k\rangle$. The reason is because of level matching, see Eq. (21).

To summarize: open string has a tachyon, a photon and so on; closed string has a tachyon, a graviton, dilaton, axion and so on. 'so on' here denotes massive states, meaning they have m^2 of the order of $1/\alpha'$.