Physics G8099/V3400: String Theory
Instructors: Lam Hui and Dan Kabat

Here are some notes on superstring. Embedded in here are some problems that I encourage you to do. They are highlighted in boldface.

The treatment follows Polchinski's rather closely. The aim is to fill in some of the missing steps.

Action.

Let us consider the following action

$$S = -\frac{1}{4\pi}\int d\tau d\sigma \frac{1}{\alpha'}[-\partial_\tau X \partial_\tau X + \partial_\sigma X \partial_\sigma X] + i\varphi^\dagger \Gamma^0 \Gamma^a \partial_a \varphi \tag{1}$$

This is in conformal gauge. The first term gives the bosonic Lagrangian. The second term is the Dirac Lagrangian for a 2 component massless spinor φ on the worldsheet. Note that the index a is a worldsheet index. One can check that this agrees with the usual Dirac Lagrangian in your field theory textbook by noting that $\Gamma^a = -i\gamma^a$; the extra factor of 4π is chosen to conform to Polchinski's notation.

For $D = 2$, we have for Dirac representation:

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

Note also that in $D = 2$, charge conjugation can be defined simply as $\varphi_c = \varphi^*$ (i.e. the matrix B that we defined in class equals the identity). We will assume φ is Majoranna i.e. real, and write its two components as φ_+ and φ_-. Hence

$$S = \frac{1}{4\pi}\int d\tau d\sigma \frac{-1}{\alpha'}[-\partial_\tau X \partial_\tau X + \partial_\sigma X \partial_\sigma X] + i[\varphi_-(\partial_\tau + \partial_\sigma)\varphi_- + \varphi_+(\partial_\tau - \partial_\sigma)\varphi_+] \tag{3}$$

Euclideanize and complexify as before, i.e. define $z = \sigma - \tau = \sigma^1 + i\sigma^2$, and $\bar{z} = \sigma + \tau = \sigma^1 - i\sigma^2$. Define also $\psi = (-i)^{1/2}\varphi_-$, $\tilde{\psi} = i^{1/2}\varphi_+$, and recall that $iS = -S_E$, we have

$$S_E = \frac{1}{4\pi}\int d^2 z \frac{2}{\alpha'}\partial X \bar{\partial} X + \psi \bar{\partial}\psi + \tilde{\psi}\partial\tilde{\psi} \tag{4}$$

where $d^2 z = dz d\bar{z}$. Here, X, ψ and $\tilde{\psi}$ all carry a spacetime index which has been suppressed, and will be suppressed for the most part in the rest of the notes.

Symmetries.

The above has the usual conformal symmetry, where $z \to z' = f(z)$ with f being a holomorphic function of z. Under this, we have $X \to X'(z', \bar{z}') = X(z, \bar{z})$, $\partial X \to \partial' X'(z') = (\partial z/\partial z')\partial X(z)$, $\psi \to \psi'(z') = (\partial z/\partial z')^{1/2}\psi(z)$, $\tilde{\psi} \to \tilde{\psi}'(\bar{z}') = (\partial\bar{z}/\partial\bar{z}')^{1/2}\tilde{\psi}(z)$. Note that while X is a scalar, ψ and $\tilde{\psi}$ are not: they transform with weight 1/2, as if they carry half an index (∂X on the other hand transforms with weight 1 as it should). It is easy to check that S_E is invariant under the above transformations.

It is useful to give the infinitesimal form of the conformal transformation. By this we mean $z' = z + v(z)$, where v is assumed to be small. We can think of transforming X, ψ and $\tilde{\psi}$ (at the same point z i.e. without moving z around) as

$$\delta X = -v\partial X - v^*\bar{\partial}X \quad , \quad \delta\psi = -\frac{1}{2}(\partial v)\psi - v\partial\psi \quad , \quad \delta\tilde{\psi} = -\frac{1}{2}(\bar{\partial}v^*)\tilde{\psi} - v^*\bar{\partial}\tilde{\psi} \tag{5}$$

Problem 1. Verify that the above is indeed a symmetry of S_E i.e. $\delta S_E = 0$. The key is to remember that $\bar{\partial}v = \partial v^* = 0$.

It turns out S_E also has a superconformal symmetry, whose associated transformations are:

$$\delta X = \left(\frac{\alpha'}{2}\right)^{1/2}(\eta\psi + \eta^*\tilde{\psi}) \quad , \quad \delta\psi = -\left(\frac{2}{\alpha'}\right)^{1/2}\eta\partial X \quad , \quad \delta\tilde{\psi} = -\left(\frac{2}{\alpha'}\right)^{1/2}\eta^*\bar{\partial}X \tag{6}$$

where $\eta = \eta(z)$ is a holomorphic Grassmann function. We have verified in class that this is a good symmetry. The crucial point is that $\bar{\partial}\eta = \partial\eta^* = 0$.

Problem 2. Show that commuting two superconformal transformations give a conformal transformation. Proceed as follows. Consider

$$\delta_{\eta_1}\psi = -\left(\frac{2}{\alpha'}\right)^{1/2}\eta_1\partial X \quad , \quad \delta_{\eta_2}\delta_{\eta_1}\psi = -\left(\frac{2}{\alpha'}\right)^{1/2}\eta_1\partial\delta_{\eta_2}X = -\eta_1\partial(\eta_2\psi + \eta_2^*\tilde{\psi}) \tag{7}$$

Show that $[\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1}]\psi = \delta_v\psi$, where $\delta_v\psi$ is the conformal transformation (Eq. [5]) with $v = 2\eta_2\eta_1$. Along the way, you would need to assume the equation of motion $\partial\tilde{\psi} = 0$. (It is a little odd that such a symmetry statement holds only on equation of motion. It turns out with more formalism, it can be shown that the same statement holds even away from equation of motion.)

Currents.

To derive the conserved currents associated with the above symmetries, we will proceed as follows. Let's consider the conformal symmetry as an example. Imagine modifying the transformation in Eq. (5) by $v \to v\rho$ and $v^* \to v^*\rho$, where ρ is a real function of z and $\bar{z}$. With this modification, the transformation no longer constitute a symmetry of the action, unless ρ is simply a constant (this is because $v\rho$ is not holomorphic and $v^*\rho$ is not antiholomorphic). Therefore, instead of $\delta S_E = 0$, we expect

$$\delta S_E = -\frac{i}{2\pi}\int dzd\bar{z}[j_z\bar{\partial}\rho + j_{\bar{z}}\partial\rho] \tag{8}$$

The factors of i and 2π are just (Polchinksi's) convention. The important point is that this vanishes if ρ is just a constant. Since ρ is arbitrary, you can see that integrating the above by parts, $\bar{\partial} j_z + \partial j_{\bar{z}} = 0$ on the equation of motion i.e. j_z and $j_{\bar{z}}$ are components of the conserved current.

Let's practice by deriving the conserved current corresponding to conformal symmetry. Let's consider the bosonic part first:

$$\delta S_E^X = -\frac{2}{2\pi\alpha'} \int d^2z [\partial(\rho v \partial X + \rho v^* \bar{\partial} X) \bar{\partial} X] \tag{9}$$

where the factor of 2 takes care of the fact that δX can show up in both ∂X part or $\bar{\partial} X$ part, but one can exchange them via integration by parts. Using the fact that $\bar{\partial} v = \partial v^* = 0$, and integrating by parts, we find

$$\begin{aligned} \delta S_E^X &= -\frac{1}{\pi\alpha'} \int d^2z [v \bar{\partial}(\rho \partial X) \partial X + v^* \partial(\rho \bar{\partial} X) \bar{\partial} X] \\ &= -\frac{1}{\pi\alpha'} \int d^2z [v \bar{\partial}\rho \partial X \partial X + v^* \partial \rho \bar{\partial} X \bar{\partial} X + v\rho(\bar{\partial}\partial X)\partial X + v^* \rho(\partial \bar{\partial} X) \bar{\partial} X] \\ &= -\frac{1}{2\pi\alpha'} \int d^2z [v \bar{\partial}\rho \partial X \partial X + v^* \partial \rho \bar{\partial} X \bar{\partial} X] \end{aligned} \tag{10}$$

where we have used $v\rho(\bar{\partial}\partial X)\partial X = (1/2) v\rho \bar{\partial}(\partial X \partial X) = -(1/2) v (\bar{\partial}\rho)(\partial X \partial X)$, and $v^*\rho(\partial\bar{\partial} X)\bar{\partial} X = (1/2) v^* \rho \partial(\bar{\partial} X \bar{\partial} X) = -(1/2) v^* (\partial\rho)(\bar{\partial} X \bar{\partial} X)$, liberally integrating by parts (i.e. equalities here mean up to total derivatives).

Similarly, the fermionic part:

$$\begin{aligned} \delta S_E^\psi &= -\frac{2}{4\pi} \int d^2z [(\frac{1}{2}\partial(v\rho)\psi + v\rho\partial\psi)\bar{\partial}\psi] \\ &= -\frac{1}{2\pi} \int d^2z [\frac{1}{2}\partial(v\rho)\psi\bar{\partial}\psi + \psi\bar{\partial}(v\rho\partial\psi)] \\ &= -\frac{1}{2\pi} \int d^2z [\frac{1}{2}\partial(v\rho)\psi\bar{\partial}\psi + \bar{\partial}(v\rho)\psi\partial\psi + v\rho\psi\bar{\partial}\partial\psi] \\ &= -\frac{1}{4\pi} \int d^2z\, v(\bar{\partial}\rho)\psi\partial\psi \end{aligned} \tag{11}$$

where we have made use of the fact that $2v\rho\psi\bar{\partial}\partial\psi = -v(\bar{\partial}\rho)\psi\partial\psi - \partial(v\rho)\psi\bar{\partial}\psi$ (up to total derivatives).

We have also:

$$\delta S_E^{\tilde{\psi}} = -\frac{1}{4\pi} \int d^2z\, v^*(\partial\rho)\tilde{\psi}\bar{\partial}\tilde{\psi} \tag{12}$$

Finally, collecting all three δS_E's, and compare against Eq. (8), we can see that

$$i j_z = \frac{1}{\alpha'} v \partial X \partial X + \frac{1}{2} v \psi \partial \psi \quad , \quad i j_{\bar{z}} = \frac{1}{\alpha'} v^* \bar{\partial} X \bar{\partial} X + \frac{1}{2} v^* \tilde{\psi} \bar{\partial} \tilde{\psi} \tag{13}$$

Since v is arbitary, it is convenient to define currents that are independent of v. Again, following Polchinski's definition:

$$j_z \equiv ivT^B \quad , \quad j_{\bar{z}} \equiv iv^* \tilde{T}^B \tag{14}$$

where T^B and $\tilde{T}^B$ should be thought of as the two components of the energy momentum tensor: T^B_{zz} and $T^B_{\bar{z}\bar{z}}$. Here, B denotes bosonic, since T^B and $\tilde{T}^B$ are bosonic. We therefore have

$$T^B = -\frac{1}{\alpha'}\partial X \partial X - \frac{1}{2}\psi\partial\psi \quad , \quad \tilde{T}^B = -\frac{1}{\alpha'}\bar{\partial} X \bar{\partial} X - \frac{1}{2}\tilde{\psi}\bar{\partial}\tilde{\psi} \tag{15}$$

The X parts of these agree with the energy momentum tensor we derived earlier for bosonic string.

Problem 3. Derive the conserved current and therefore associated 'energy momentum' tensor for superconformal transformation. Here's how you can proceed. Consider the following modified version of Eq. (6):

$$\delta X = \left(\frac{\alpha'}{2}\right)^{1/2}(\eta\rho\psi + \eta^*\rho\tilde{\psi}) \quad , \quad \delta\psi = -\left(\frac{2}{\alpha'}\right)^{1/2}\eta\rho\partial X \quad , \quad \delta\tilde{\psi} = -\left(\frac{2}{\alpha'}\right)^{1/2}\eta^*\rho\bar{\partial} X \tag{16}$$

Work out δS_E, and use Eq. (8) to show that

$$ij_z = -\left(\frac{2}{\alpha'}\right)^{1/2}\eta\psi\partial X \quad , \quad ij_{\bar{z}} = -\left(\frac{2}{\alpha'}\right)^{1/2}\eta^*\tilde{\psi}\bar{\partial} X \tag{17}$$

Following Polchinski to define the fermionic 'energy momentum' tensor:

$$j_z = \eta T^F \quad , \quad j_{\bar{z}} = \eta^* \tilde{T}^F \tag{18}$$

show that

$$T^F = i\left(\frac{2}{\alpha'}\right)^{1/2}\psi\partial X \quad , \quad \tilde{T}^F = i\left(\frac{2}{\alpha'}\right)^{1/2}\tilde{\psi}\bar{\partial} X \tag{19}$$

Green function.

We have derived in class:

$$\langle\psi^\mu(z)\psi^\nu(z')\rangle = \frac{\eta^{\mu\nu}}{z - z'} \quad , \quad \langle\tilde{\psi}^\mu(\bar{z})\tilde{\psi}^\nu(\bar{z}')\rangle = \frac{\eta^{\mu\nu}}{\bar{z} - \bar{z}'} \tag{20}$$

Note that the equations of motion $\bar{\partial}\psi^\mu = 0$ and $\partial\tilde{\psi}^\mu = 0$ imply that ψ^μ is holomorphic and $\tilde{\psi}^\mu$ is antiholomorphic.

Boundary conditions.

In a lot of the manipulations above, we liberally integrate by parts and throw out boundary terms. This is justified only if the string has the appropriate boundary conditions. We have already considered the open and closed string boundary conditions for the bosonic part. Let focus on the fermionic part. Consider the variation of the action in Eq. (3) under variation in the fermionic components. The surface term is:

$$\delta S \propto [\varphi_- \delta\varphi_- - \varphi_+ \delta\varphi_+]_{\sigma=0}^{\sigma=\ell} \tag{21}$$

The closed string boundary condition is to demand that φ_- and φ_+ are either periodic or anti-periodic so that the above surface term vanishes. Recalling that ψ and $\tilde{\psi}$ are simply proportional to φ_- and φ_+, this means:

$$\psi(\sigma = 0) = \pm\psi(\sigma = \ell) \quad , \quad \tilde{\psi}(\sigma = 0) = \pm\tilde{\psi}(\sigma = \ell) \quad , \tag{22}$$

The positive sign is called Ramond (R), and negative is Neveu-Schwarz (NS). For the closed string, it is natural to adopt the convention $\ell = 2\pi$.

The open string boundary condition is one where the surface term vanishes because the contributions at $\sigma = \ell$ and at $\sigma = 0$ vanish separately by themselves. This requires

$$\psi(\sigma = 0) = \pm\tilde{\psi}(\sigma = 0) \quad , \quad \psi(\sigma = \ell) = \pm\tilde{\psi}(\sigma = \ell) \tag{23}$$

By field redefinition, one can always choose $\psi(\sigma = \ell) = +\tilde{\psi}(\sigma = \ell)$. Therefore, one is again left with two possibilities:

$$\psi(\sigma = 0) = \pm\tilde{\psi}(\sigma = 0) \tag{24}$$

where $+$ is Ramond, and $-$ is NS.

For open string, we will adopt the convention $\ell = \pi$, following Polchinski. However, there is a way, known as the doubling trick, that kind of extends the open string beyond π to make it look as much like the closed string as possible. Here's how it works. It is easiest to think in terms of the complex coordinate $z = e^{-i(\sigma^1 + i\sigma^2)}$. (Recall that by conformal symmetry, we can freely go from our earlier $z = \sigma^1 + i\sigma^2$ to this z.) For the open string, σ^1 goes from 0 to π, which means z covers the lower half of the complex plane. The field $\psi(z)$ is defined only for z belonging to the lower half plane. The field $\tilde{\psi}(\bar{z})$, on the other hand, is defined only for $\bar{z}$ belonging to the upper half plane. Let us therefore define an extension of ψ into the upper half plane as follows: suppose z_0 belongs to the upper half plane, $\psi(z_0)$ is defined to be $\tilde{\psi}(\bar{z} = z_0)$ (in other words, this is the value of $\tilde{\psi}$ at $z = \bar{z}_0$). This allows us then to have a ψ that is defined over the entire complex plane. For $\sigma^1 = 0$ to π, ψ is just defined as usual. For $\sigma^1 = \sigma_0 > \pi$, ψ basically takes on the value of $\tilde{\psi}$ at $\sigma^1 = 2\pi - \sigma_0$. (This is nicely consistent with our open string boundary condition above, where ψ is equal to $\tilde{\psi}$ at π.) With this doubling trick, the 2 possible open string boundary conditions become:

$$\psi(\sigma = 0) = \pm\psi(\sigma = 2\pi) \tag{25}$$

which looks just like the closed string boundary conditions. The only difference between open and closed strings is that we need to worry about only the left movers for open string, while both left and right movers need to be dealt with for closed string.

Mode expansions.

Consider first mode expansion for ψ. Let us use w to denote $\sigma^1 + i\sigma^2$. The R and NS boundary conditions can be represented by

$$\psi(w + 2\pi) = e^{2\pi i\nu}\psi(w) \quad , \quad \tilde{\psi}(w + 2\pi) = e^{-2\pi i\tilde{\nu}}\tilde{\psi}(w) \tag{26}$$

where ν, $\tilde{\nu}$ equal 0 for R, and 1/2 for NS. The minus sign for $\tilde{\psi}$ is such that we get similar looking mode expansions for the left and right movers later on. Remember that here, as in the rest of the notes, left mover expressions apply equally well to open and closed strings, while right mover expressions apply to closed string.

Define the mode expansion:

$$\psi(w) = i^{-1/2} \sum_{r=Z+\nu} \psi_r e^{irw} \quad , \quad \tilde{\psi}(\bar{w}) = i^{1/2} \sum_{r=Z+\tilde{\nu}} \tilde{\psi}_r e^{-ir\bar{w}} \tag{27}$$

Here, Z is supposed to denote integers.

A conformal transformation from w to $z = e^{-iw}$ gives us

$$\psi \rightarrow \psi(z) = (\partial_z w)^{1/2}\psi(w) = i^{1/2} z^{-1/2}\psi(w) \tag{28}$$

This means the mode expansion for $\psi(z)$ looks particularly simple:

$$\psi(z) = \sum_{r=Z+\nu} \frac{\psi_r}{z^{r+\frac{1}{2}}} \quad , \quad \tilde{\psi}(\bar{z}) = \sum_{r=Z+\tilde{\nu}} \frac{\tilde{\psi}_r}{\bar{z}^{r+\frac{1}{2}}} \quad , \tag{29}$$

Inverting these is also easy:

$$\psi^\mu_r = \oint \frac{dz}{2\pi i} z^{r-\frac{1}{2}} \psi^\mu(z) \quad , \quad \tilde{\psi}^\mu_r = -\oint \frac{d\bar{z}}{2\pi i} \bar{z}^{r-\frac{1}{2}} \tilde{\psi}^\mu(\bar{z}) \quad , \tag{30}$$

Problem 4. Using the Green function for ψ, derive the anticommutator:

$$\{\psi^\mu_r, \psi^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s} \tag{31}$$

You might find it useful to recall how we derived the commutator for the oscillators of the bosonic string. Very similar contour arguments work here. Keep in mind that ψ is Grassmann.

Let me collect here the mode expansions for X just to be complete:

$$\partial X = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_n \frac{\alpha_n}{z^{n+1}} \quad , \quad \bar{\partial} X = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_n \frac{\tilde{\alpha}_n}{\bar{z}^{n+1}} \tag{32}$$

Similarly, the mode expansions for T_B (Eq. [15]) and T_F (Eq. [19]) are defined to be

$$T_B = \sum_m \frac{L_m}{z^{m+2}} \quad , \quad T_F = \sum_{r=Z+\nu} \frac{G_r}{z^{r+\frac{3}{2}}} \tag{33}$$

There are as usual the antiholomorphic counterparts for the closed string.

Problem 5. Using the Green functions $\langle XX \rangle$ and $\langle \psi\psi \rangle$, show that the superstring energy momentum oscillators satisfy the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \tag{34}$$

where c the central charge now equals $D + D/2 = 3D/2$, where the $D/2$ comes from the fermions. The argument should proceed in an analogous way to what we did in the class for the bosonic string. It is important to keep in mind that our definition for T_B in Eq. (15) is implicitly normal ordered in the sense that we subtract out expectation values of products of operators. To be precise:

$$T^B = -\frac{1}{\alpha'}(\partial X \partial X - \langle \partial X \partial X \rangle) - \frac{1}{2}(\psi \partial \psi - \langle \psi \partial \psi \rangle) \tag{35}$$

It turns out ghosts in the theory contribute a central charge of -15. Requiring the total central charge to vanish therefore demands $3D/2 - 15 = 0$ i.e. $D = 10$. Vanishing total central charge guarantees that conformal symmetry is respected in the quantum theory. (In the case of the bosonic string, the ghost central charge turns out to be -26, and there the total central charge is therefore $D - 26$, which is why we required $D = 26$. Why the ghosts should give these central charge values is something I won't go into here.)

I will leave without proof two further (anti-) commutators:

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r,-s} \tag{36}$$

$$[L_m, G_r] = \frac{m-2r}{2} G_{m+r} \tag{37}$$

Finally, one can relate G_r and L_m to the X and ψ oscillators by plugging Eqs. (29) and (32) into Eq. (33):

$$G_r = \sum_{n=Z} \alpha_n \psi_{r-n} \tag{38}$$

$$L_m = a\delta_{m,0} + \frac{1}{2}\sum_n : \alpha_{m-n}\alpha_n : + \frac{1}{4}\sum_{r=Z+\nu} (2r-m) : \psi_{m-r}\psi_r : \tag{39}$$

where the colons mean we put creation operators to the left and annihilation operators to the right. The term a is an ordering constant that exists for $m = 0$, and is equal to $D/16$ for R and 0 for NS. One way to see it is to let $[L_1, L_{-1}] = 2L_0$ act on the NS and R ground states with $k^\mu = 0$. I won't do this here. For later convenience, let me define the part of L_m that is free of a as:

$$L_m^{\text{norm}} = \frac{1}{2}\sum_n :\alpha_{m-n}\alpha_n: + \frac{1}{4}\sum_{r=Z+\nu}(2r-m):\psi_{m-r}\psi_r: \tag{40}$$

For working out the spectrum, it is useful to write down explicit expressions for the low lying L_m's and G_r's:

$$\begin{aligned}
\text{NS}: \quad L_0^{\text{norm}} &= \frac{1}{2}\alpha_0^2 + \alpha_{-1}\alpha_1 + ... + \frac{1}{2}\psi_{-\frac{1}{2}}\psi_{\frac{1}{2}} + ... \qquad (41)\\
\text{NS}: \quad L_1 &= \frac{1}{2}(\alpha_0\alpha_1 + \alpha_{-1}\alpha_2 + ...) + \psi_{-\frac{1}{2}}\psi_{\frac{3}{2}} + ...\\
\text{NS}: \quad L_{-1} &= \frac{1}{2}(\alpha_{-1}\alpha_0 + \alpha_{-2}\alpha_1 + ...) + \psi_{-\frac{3}{2}}\psi_{\frac{1}{2}} + ...\\
\text{NS}: \quad G_{\frac{1}{2}} &= \alpha_0\psi_{\frac{1}{2}} + \alpha_1\psi_{-\frac{1}{2}} + \alpha_{-1}\psi_{\frac{3}{2}} + ...\\
\text{NS}: \quad G_{-\frac{1}{2}} &= \alpha_0\psi_{-\frac{1}{2}} + \alpha_1\psi_{-\frac{3}{2}} + \alpha_{-1}\psi_{\frac{1}{2}} + ...\\
\text{R}: \quad L_0^{\text{norm}} &= \frac{1}{2}\alpha_0^2 + \alpha_{-1}\alpha_1 + ... + \psi_{-1}\psi_1 + ...\\
\text{R}: \quad L_1 &= \frac{1}{2}(\alpha_0\alpha_1 + \alpha_{-1}\alpha_2 + ...) + \frac{1}{2}\psi_0\psi_1 + ...\\
\text{R}: \quad L_{-1} &= \frac{1}{2}(\alpha_{-1}\alpha_0 + \alpha_{-2}\alpha_1 + ...) + \frac{1}{2}\psi_{-1}\psi_0 + ...\\
\text{R}: \quad G_0 &= \alpha_0\psi_0 + \alpha_1\psi_{-1} + \alpha_{-1}\psi_1 + ...\\
\text{R}: \quad G_1 &= \alpha_0\psi_1 + \alpha_1\psi_0 + \alpha_{-1}\psi_2 + ...\\
\text{R}: \quad G_{-1} &= \alpha_0\psi_{-1} + \alpha_1\psi_{-2} + \alpha_{-1}\psi_0 + ...
\end{aligned}$$

Spectrum.

Finally, we are ready to discuss the spectrum. We will impose the physical state conditions:

$$\begin{aligned}
&L_m^{\text{norm}} + A\delta_{m,0}|\phi\rangle = 0 \quad , \quad m > 0 \qquad (42)\\
&G_r|\phi\rangle = 0 \quad , \quad r > 0
\end{aligned}$$

Here, A is the ordering constant, but I purposefully choose the symbol A instead of a as in Eq. (39) because it turns out ghosts modify the constant to a different number. I will state without proof that $A = 0$ for R and $A = -1/2$ for NS. This is the analog of $A = -1$ for the bosonic string. There is a useful mneumonic for working out the appropriate values of A in each case. In the bosonic string, $A = (26-2) \times (-1/24) = -1$. Here, 26 is the number of X's we start out with. Ghosts remove 2 of these. Each bosonic degree of freedom gives a factor of $-1/24$. In the R superstring, $A = (10-2)\times(-1/24+1/24) = 0$. Here, 10 is the number of X's and ψ's we start out with. Ghosts remove 2 of these. Each periodic fermionic degree of freedom gives $+1/24$. Each antiperiodic fermionic degree of freedom gives $-1/48$ instead, which is why in the NS superstring, $A = (10-2)\times(-1/24-1/48) = -1/2$. All this is nothing more than mneumonics, though it's possible to justify it in a rigorous way.

Spurious states are states of the form:

$$\begin{aligned} L_n|\chi\rangle \quad &, \quad n<0 \\ G_r|\chi\rangle \quad &, \quad r<0 \end{aligned} \tag{43}$$

where $|\chi\rangle$ is some arbitrary state.

Null states are both spurious and physical and should be mod out.

Applying these rules give the states listed in Table 10.2 of Polchinski for open string, and Table 10.3 for closed string. Applying the GSO projection gives us the IIA and IIB superstrings discussed on p. 26 and 27. It turns out these (oriented) theories cannot be consistently coupled to open strings so IIA and IIB are closed supertring theories. Type I theory is where one loses one of the gravitinos by projecting out parity odd states; this theory does allow for open string, and so type I is an unoriented open plus closed superstring theory. You are encouraged to see if you can reproduce Tables 10.2 and 10.3, using perhaps your class notes as a guide.