Astronomy C3602: Homework #3

Due in class on Wednesday, March 5, 2008

Problem 1: Seeing Around the Universe (30 points).

A photon is emitted at the time of the Big Bang in a universe that contains only non-relativistic matter, and has $k > 0$. Here we will show that the photon travels precisely all the way around the universe by the time of the "Big Crunch".

(a) Recall that the spatial part of the metric

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + R_0^2 \sin^2(r/R_0)(d\theta^2 + \sin^2\theta d\phi) \right] \tag{1}$$

for a positively curved space-time describes a sphere of radius R_0, where the radius of curvature R_0 is related to k by $k = c^2/R_0^2$. What is the proper circumference of the universe at cosmic time t? **[10 points]**

(b) Choose the coordinate system such that the photon is emitted at $t = 0$ at the origin $r = 0$, and travels along the geodesic with $\theta = \phi = 0$. Write down the expression for its proper distance from the origin at cosmic time t as an integral over t. Divide this expression by your answer in (a) to obtain the fraction $f(t)$ of the circumference covered by the photon by time t. **[10 points]**

(c) Use the parametric solution given in Problem Set #2,

$$a(\theta) = \frac{4\pi G\rho_0}{3k}(1 - \cos\theta)$$
$$t(\theta) = \frac{4\pi G\rho_0}{3k^{3/2}}(\theta - \sin\theta)$$

to convert the integral over t in (b) to an integral over θ, and find the function $f(\theta)$. Show that $f(\theta = \pi) = 1/2$ and $f(\theta = 2\pi) = 1$. **[10 points]**

Problem 2: The Age of our Universe (30 points).

Consider a flat universe with $k = 0$ that has non-relativistic matter and a cosmological constant, with $\Omega_{\Lambda,0} + \Omega_{m,0} = 1$ (i.e., we will ignore radiation which affects the evolution only at very early times, and has only a small effect on the present age).

(a) Show that the Friedmann equation can be written as

$$\dot{a}^2 = H_0^2 \left[\frac{\Omega_{m,0}}{a} + (1 - \Omega_{m,0})a^2 \right] \tag{2}$$

[5 points]

(b) Solve the above equation to obtain an explicit relation between the cosmic time t and redshift $(1+z) \equiv 1/a$. Hint: you will need the indefinite integral $\int dx(a+bx^2)^{-1/2} = b^{-1/2}\ln[x\sqrt{b}+\sqrt{a+bx^2}]$. **[20 points]**

(c) What is the current age if $\Omega_{m,0} = 0.3$, and H_0 =70 km/s/Mpc? What was the age of the universe when the light from the most distant known object (a galaxy at redshift $z = 6.6$) was emitted? **[5 points]**

Problem 3: A Dark Matter "Model" (30 points).

Assume that the dark matter is composed of a population of iron asteroid balls that fill space uniformly. Each iron ball is identical and has an internal density $\rho_0 = 8$ g cm^{-3}, and a radius r. For simplicity, in this problem, assume a matter dominated universe with $\Omega_{m,0} = 1$ and H_0 =70 km/s/Mpc.

(a) What fraction of the sky would be covered by the collection of asteroids between us ($z = 0$) and a source at redshift z_s ? Your answer will depend explicitly on r. (Hint: this is similar to problem 3(a) in Problem Set #1, except that both the space density of objects, and the volume per unit redshift, evolves.) **[20 points]**

(b) What is the lower limit on r from the condition that the universe is not opaque; i.e. that we can see a quasar at redshift $z_s = 5$? **[10 points]**

Problem 4: Dark Matter - 1933 vs. now (10 points).

Zwicky (1933) compared the mass–to–light (M/L) ratio of the Coma cluster as measured from the virial theorem with the M/L ratio of the luminous parts of individual galaxies as measured by rotation curves. He concluded that there was 400 times as much dark matter as luminous matter in the Coma cluster. However, Zwicky's conclusion was based on the erroneous value of the Hubble constant, H_0 =558 km/s/Mpc. How is Zwicky's conclusion about the ratio of dark to luminous matter affected, now that we know that the correct value of the Hubble constant is much smaller, H_0 =70 km/s/Mpc? **[10 points]**